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ON THE HYBRID GAS LUBRICATED JOURNAL BEARING

by

J.W. Lund

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ABSTRACT

The purpose of this report is to analyze the load carrying capacity of an external pressurized gas journal bearing including the hydrodynamic effect caused by the rotational speed of the journal. The analysis is an approximate first order pertubation solution, i.e. it assumes small bearing eccentricity ratio. Furthermore, it is assumed that the gas feeding takes place through orifices in the centerplane of the bearing and that the number of feeding holes is sufficiently large to be considered a line source.

Results are given for the load carrying capacity and for the attitude angle. In addition an investigation is made of the validity of the approximation used in arriving at the solution.

INTRODUCTION

Gas journal bearings are principally of two types: 1) self acting, and 2) externally pressurized. The self acting bearing works by hydrodynamic action and, therefore, has limited load capacity. It requires a small clearance and tight manufacturing tolerances. The externally pressurized bearing relies on hydrostatic pressure to generate its load carrying capacity. It is used in applications with higher load requirements and with less manufacturing precision but requires an external pressure supply.

Although the externally pressurized bearing is primarily hydrostatic, the rotation of the journal causes some hydrodynamic action. This effect is in most cases negligible because of the relatively large clearance but may become significant at higher speeds and at large eccentricity ratio. Since in recent times such cases have been encountered it is of value to develop an analysis for the combined hydrostatic-hydrodynamic gas journal bearing. This is the purpose of the present report.

RESULTS

The results of the analysis are presented in Figures 1-12 where curves for load carrying capacity and attitude angle are given. Four dimensionless numbers are necessary for a bearing calculation:

the bearing number $\Lambda = \frac{6\mu\omega}{P_a}\left(\frac{R}{C}\right)^2$

the bearing length-to-diameter ratio = L/D

the feeding parameter $\Lambda_t = \frac{6\mu N a^2 \alpha}{P_a C^3} \cdot \sqrt{\frac{RT}{m}}$

the pressure ratio = P_{supply} / P_{amb}

An estimate of the validity of the results can be obtained from Figures 21-28 where the dotted curves are the previous results and the solid curves present an improved analysis. The improved analysis, however, only satisfies the analysis within a certain error which is marked on the curves. Thus a comparison between the two analyses is only valid when the error is small.

DISCUSSION OF ANALYSIS

Theoretical analyses exist separately for the self acting bearing and for the externally pressurized bearing under the assumption that the eccentricity ratio ε is small (Ref. 2 and 3). Both these solutions are first order pertubation solutions around $\varepsilon = 0$ such that the bearing load becomes proportional to ε. The results are in good agreement with more extensive solutions and with experimental results. Thus it seems logical to attempt an analysis that combines the two solutions.

Assuming isothermal conditions in the gas film the exact equation for the problem under consideration is Reynolds' equation:

$$\frac{\partial}{\partial x}\left[\rho h^3 \frac{\partial P}{\partial x}\right] + \frac{\partial}{\partial z}\left[\rho h^3 \frac{\partial P}{\partial z}\right] = 6\mu R\omega \frac{\partial(\rho h)}{\partial x} - 12\mu m$$

where P = film pressure, ρ = gas density, h = film thickness, R = journal radius, ω = angular speed of journal, μ = gas viscosity, x = circumferential coordinate, z = axial coordinate and m = mass flow per square inch from the external pressure source (m is a function of the local pressure P). This mass flow enters the equation only at the discreet points at which flow is introduced. For this reason it is extremely difficult to obtain a solution of the above equation even in approximate form. To circumvent this difficulty an assumption shall be made in regard to the nature of the gas feeding. The gas feeding takes place through orifice restricted feeding holes in the center-plane of the bearing. It shall be assumed that the number of feeding holes is sufficiently large to be treated as a line source. Thus the gas feeding no

longer enters the differential equation but instead becomes a boundary condition to the equation. This boundary condition stipulates that at the centerplane of the bearing the pressure gradient must be such that the flow into the bearing is equal to the outflow from the feeding holes. The flow through the feeding holes is restricted by orifices. Any restriction due to the annular flow area existing between the edge of the feeding hole and the journal surface is neglected.

The normal equation for flow through an orifice has a discontinuity when the pressure ratio reaches a critical value. This discontinuity is eliminated by using a modified orifice flow equation which has proven to give satisfactory results.

Based on the above assumptions the equation defining the gas feeding can be set up. It is found to be governed by the dimensionless feeding parameter:

$$\Lambda_t = \frac{6\mu N \alpha a^2}{P_a C^3} \cdot \sqrt{\frac{\bar{R}T}{m}}$$

(μ = viscosity, P_a = ambient pressure, C = radial clearance, N = number of feeding holes, α = orifice coefficient, a = orifice radius, m = number of orifices in series, $\bar{R}$ = gas constant, T = total temperature.)

and the ratio of supply pressure to ambient pressure:

$$V = \frac{P_s}{P_a}$$

Returning to Reynolds' equation it is written in dimensionless form disclosing two additional dimensionless parameters: the bearing number

$$\Lambda = \frac{6\mu\omega}{P_a}\left(\frac{R}{C}\right)^2$$

(ω = angular journal speed, R = journal radius)

and the bearing length-to-diameter ratio:

$$\xi = \frac{L}{D}$$

The solution will be a function of these 4 parameters. A first order pertubation solution of Reynolds' equation is desired.

Thus the film pressure is approximated by:

$$P = P_0 + \varepsilon P_1$$

P_o is the pressure distribution when the journal is concentric in the bearing, and is found to be:

$$P_0 = \sqrt{1 + \rho^2(\xi - \zeta)}$$

where ρ^2 is a constant determined by the bearing-feeding system and $(\xi - \zeta)$ is the axial distance measured from the end of the bearing, i.e. P_o^2 is a linear function. Substituting the above first order expression for the pressure into Reynolds' equation yields the first order pertubation equation with boundary conditions, see Appendix, eq. 5:

$$\frac{\partial^2(P_0P_1)}{\partial\theta^2} + \frac{\partial^2(P_0P_1)}{\partial\zeta^2} - \frac{\Lambda}{P_0}\frac{\partial(P_0P_1)}{\partial\theta} = -\Lambda P_0 \sin\theta$$

Unfortunately this equation has no simple solution due to the form of P_o. Instead the nature of the solution is explored. This is done by approximating $\frac{\Lambda}{P_0}$ by a constant:

$$\frac{\Lambda}{P_0} = \frac{\Lambda}{P_{o,av}}$$

where $P_{o,\,av}$ is the average value of P_o. Then the equation is similar to Ref. 1 and 2 and a solution can be obtained. Numerical values of the dimensionless load carrying capacity and the attitude angle are given in Figs. 1-12 as a function of Λ for a wide range of Λ_t-values, two L/D-ratios and

three pressure ratios.

The second part of the analysis evaluates the validity of the approximation Λ_t = constant and establishes the regions in which the results are valid. Whereas previously Λ_t was constant throughout the bearing, it now varies along the length of the bearing. Since P_o^2 is linear it is reasonable to replace Λ_t with a linear or almost linear expression. This leads to a solution involving Bessel functions of a complex variable which is rather impractical to evaluate and instead the Bessel functions are approximated by complex trigonometric functions. Hence the solution does not satisfy the differential equation exactly and the discrepancy is presented in Figures 13-20. This error must be small compared to 1. Otherwise the analysis proceeds as before and results for load carrying capacity and attitude angle are given in Figures 21-28. In the same figures are also shown the results for Λ_t = constant for comparison. In addition the error in using the trigonometric approximation is indicated on the curves as taken from Figures 13-20. For Λ less than 1.1 the agreement is good for the load carrying capacity and reasonably good for the attitude angle. No obvious correlation can be found between the magnitude of the calculated error and the point of divergence of the results. For the particular case of L/D = 2, V = 2, and Λ_t = .5, the error never exceeds 2% and even so the results of the two analyses differ considerably. This points to the need for future investigation.

Finally, it shall be investigated if the total load carrying capacity can be

considered as a simple superposition of the hydrodynamic and the hydro-static load. Figures 21-24 indicates that this is not the case. However, since $\Lambda = \frac{6\mu\omega}{P_a}\left(\frac{R}{C}\right)^2$, where P_a can be interpreted as being the mean film pressure for a purely hydrodynamic bearing, Λ should be replaced by $\Lambda \frac{P_a}{P_{apw}}$ in applying the superposition for the hydrodynamic-hydro-static bearing. This has been done in Figures 29-32. It is seen that even with the modified Λ the superposition still is not valid. This is not surprising since the hydrodynamic and the hydrostatic action are coupled in a "non-linear" way through the feeding conditions.

CONCLUSIONS

1. The load and the attitude angle of the hybrid bearing are determined by 4 dimensionless numbers, Λ , L/D, Λ_t and P_s/P_a.

2. The rotation of the shaft increases the load carrying capacity and the attitude angle of the externally pressurized bearing. The effect is most pronounced for high values of the bearing number Λ (i. e. at high speeds), for low values of the supply pressure and for high values of the bearing length-to-diameter ratio.

3. The load carrying capacity of the hybrid bearing is not a superposition of the hydrostatic and the hydrodynamic load even if the bearing number Λ is based on the average film pressure instead of, as usual, the ambient pressure.

RECOMMENDATIONS

1. The analysis should be extended to obtain the complete solution of the first order pertubation equation.
2. An investigation of the effect of higher values of the eccentricity ratio should be undertaken.
3. The stability of the hybrid bearing should be studied especially to find out how hydrostatic pressure influences the stability of hydrodynamic bearings.

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NOMENCLATURE

Constants specifically defined in the analysis for the purpose of simplification only are not listed below.

a	Orifice radius	inch
C	Radial bearing clearance	inch
C_o, d_o	Constants given by equation (20) - (21) (1st part) and (55) - (56) (2nd part)	
F_r, F_t	Radial and tangential force components	lbs.
G	Complex form of H, see equation (12a)	
$g_1(\zeta)$, $g_2(\zeta)$	Functions defined by equation (15) - (16) (1st part) and (48) - (49) (2nd part)	
H	$= P_o P_1$	
$\bar{h}$	Film thickness	inch
h	Dimensionless film thickness $= \frac{\bar{h}}{c}$	
L	Bearing length	inch
M	Mass flow through one feedinghole	$\frac{\text{lbs. sec}}{\text{in}}$
m	Number of orifices in series in one feeding hole	
N	Number of feeding holes around circumference	
$\bar{P}$	Pressure in gas film	psia
P	Dimensionless gas film pressure $= \frac{\bar{P}}{P_a}$	
P_o	Dimensionless gas film pressure at $\varepsilon = 0$, see equation (4)	
P_1	First order term in pertubation solution, see equation (3)	
$P_{o,av}$	Average value of P_o, see equation (31)	
P_a	Ambient pressure	psia

$\overline{P}_i$	Gas film pressure at $\zeta = 0$	psia
P_s	Supply pressure	psia
ρ^2	Slope of P_o^2, see equation (9)	
q	Constant defined by equation (35)	
R	Bearing radius	inch
$\overline{R}$	Gas constant	$\dfrac{in^2}{sec^2 \cdot {}^oR}$
T	Absolute temperature	^oR
V	Pressure ratio $= \dfrac{P_s}{P_a}$	
W	Total bearing load	lbs.
X, Z	Circumferential and axial coordinates for gas film	inch
Z	Variable in 2nd part only, defined by equation (35a)	
α	Orifice coefficient	
α, β	Defined by equation (17), 1st part, and by equation (43) - (44), 2nd part	
ε	Bearing eccentricity ratio	
θ, ζ	Dimensionless circumferential and axial coordinates for gas film	
$\varkappa$	Constant defined by equation (34)	
$\varkappa_1, \varkappa_2$	Constants defined by equation (22) - (23)	
λ_1, λ_2	Constants defined by equation (57) - (58)	
Λ	Compressibility number $= \dfrac{6\mu\omega}{P_a}\left(\dfrac{R}{C}\right)^2$	
Λ_t	Feeding parameter $= \dfrac{6\mu N \alpha a^2}{P_a C^3}\sqrt{\dfrac{\overline{R}T}{m}}$	
μ	Gas viscosity	$\dfrac{lbs \cdot sec}{in^2}$

ξ	Bearing length-to-diameter ratio $= \frac{L}{2R}$	
φ	Attitude angle	
ω	Angular speed of journal	rad/sec

APPENDIX

The Combined Hydrostatic-Hydrodynamic Gas Journal Bearing
An Approximate 1st Order Pertubation Solution

The journal bearing is pressure-fed from orifice restricted feeding holes around the circumference in the centerplane of the bearing. For a sufficiently large number of feeding holes these may be approximated by a line source such that the gas feeding becomes a boundary condition to the differential equation describing the flow in the bearing (Ref. 1). The differential equation is Reynolds equation which for a perfect gas under isothermal conditions may be written:

$$\frac{\partial}{\partial x}\left[\bar{p}\bar{h}^3\frac{\partial\bar{p}}{\partial x}\right]+\frac{\partial}{\partial z}\left[\bar{p}\bar{h}^3\frac{\partial\bar{p}}{\partial z}\right]=6\mu\omega R\frac{\partial(\bar{p}\bar{h})}{\partial x}$$

To make dimensionless set:

$$\theta=\frac{x}{R}\qquad \zeta=\frac{z}{R}\qquad p=\frac{\bar{p}}{p_a}\qquad h=\frac{\bar{h}}{c}=1+\varepsilon\cos\theta$$

Then:

(1) $$\frac{\partial}{\partial\theta}\left[ph^3\frac{\partial p}{\partial\theta}\right]+\frac{\partial}{\partial\zeta}\left[ph^3\frac{\partial p}{\partial\zeta}\right]=\Lambda\frac{\partial(ph)}{\partial\theta}$$

where:

(2) $$\Lambda=\frac{6\mu\omega}{p_a}\left(\frac{R}{C}\right)^2$$

A 1st order pertubation solution with respect to the eccentricity ratio ε around $\varepsilon = 0$ shall be tried. Therefore set:

(3) $$P=P_0+\varepsilon P_1+\varepsilon^2 P_2+\cdots$$

and

$$P^2=P_0^2+2\varepsilon P_0P_1+\cdots$$

Due to symmetry P_o is a function of ζ only, i.e., $\frac{\partial P_0}{\partial\theta}=0$.

Thus the zero order equation becomes:

$$\frac{\partial^2 P_0^2}{\partial\zeta^2}=0$$

or

(4) $$P_o^2 = 1 + \mu^2(\xi - \zeta) \qquad \xi = \frac{L}{D}$$

Since $\frac{\partial^2 P_o^2}{\partial \zeta^2} = 0$ the 1st order equation becomes:

(5) $$\frac{\partial^2 H}{\partial \theta^2} + \frac{\partial^2 H}{\partial \zeta^2} - \frac{\Lambda}{P_o}\frac{\partial H}{\partial \theta} = -\Lambda P_o \sin\theta$$

where:

(6) $$H = P_o P_1$$

The boundary conditions for eq. (5) are:

<u>a) $\theta = \theta + 2\pi$</u> $\qquad H(\theta,\zeta) = H(\theta + 2\pi, \zeta)$

<u>b) $\zeta = \xi$</u> $\qquad H(\theta, \xi) = 0$

<u>c) $\zeta = 0$</u> $\qquad$ mass flow through feeding holes = mass flow through bearing

For a supply pressure of P_s, a downstream pressure of $\bar{P}_i$, gas constant $\bar{R}$, absolute temperature T, orifice radius a, orifice coefficient α and m orifices in series the mass flow M for one feeding hole is approximately:

$$M = \pi a^2 \alpha \frac{\sqrt{P_s^2 - \bar{P}_i^2}}{\sqrt{m\bar{R}T}}$$

For N feeding holes around the circumference the flow per inch is $\frac{NM}{2\pi R}$.

Equating this flow to the flow through the bearing yields:

$$-2\frac{\bar{h}^3}{12\mu}\frac{\bar{P}_i}{\bar{R}T}\left(\frac{\partial \bar{P}}{\partial \zeta}\right)_i = \frac{Na^2\alpha}{2}\frac{\sqrt{P_s^2 - \bar{P}_i^2}}{\sqrt{m\bar{R}T}}$$

Substituting dimensionless parameters gives:

$$h^3\left(\frac{\partial P^2}{\partial \zeta}\right)_i = -\Lambda_t\sqrt{V^2 - P_i^2}$$

where

(7) $$\Lambda_t = \frac{6\mu N\alpha a^2}{P_a C^3}\sqrt{\frac{\bar{R}T}{m}}$$

and

(8) $$V = \frac{P_s}{P_a}$$

Introducing eq. (3) and (4) yields ρ^2 and the boundary conditions at $\zeta = 0$:

$$\rho^2 = \Lambda_t \cdot \sqrt{V^2 - 1 - \rho^2 \zeta}$$

or

(9) $$\rho^2 = \frac{\zeta \Lambda_t^2}{2}\left[-1 + \sqrt{1 + \frac{4(V^2-1)}{\zeta^2 \Lambda_t^2}}\right]$$

and

(10) $$\left(\frac{\partial H}{\partial \zeta}\right)_i = \frac{3}{2}\rho^2 \cos\theta + \frac{\Lambda_t^2}{2\rho^3} H_i$$

<u>Approximate solution of differential equation</u>

To evaluate eq. (5) set

(11) $$H = h_1(\zeta)\sin\theta + h_2(\zeta)\cos\theta$$

which gives:

(12) $$\frac{\partial^2 G}{\partial \zeta^2} - \left(1 + i\frac{\Lambda}{P_o}\right)G = -\Lambda P_o$$

where:

(12a) $$G = h_1(\zeta) + i\, h_2(\zeta)$$

Eq. (12) is not readily solved considering that P_o is given by eq. (4). Even if an exact solution could be obtained it would be very complex and therefore be outside the scope of the present analysis which is approximate anyway. Instead the analysis will be performed in two parts. In the first part ρ is set equal to a constant. In the second part this assumption will be

investigated.

Rewrite eq. (1):

$$\frac{\partial}{\partial\theta}\left[h^3\frac{\partial P^2}{\partial\theta}\right]+\frac{\partial}{\partial\zeta}\left[h^3\frac{\partial P^2}{\partial\zeta}\right]=\frac{\Lambda}{P}\left[h\frac{\partial P^2}{\partial\theta}+P^2\frac{\partial h}{\partial\theta}\right]$$

which becomes linear in P^2 when $\frac{\Lambda}{P}$ is a known function. In order to arrive at a manageable solution we set $\frac{\Lambda}{P}$ equal to a constant, unspecified for the time being. Furthermore, only the 1st order pertubation solution is considered which is derived by substituting eq. (3):

$$\frac{\partial^2 H}{\partial\theta^2}+\frac{\partial^2 H}{\partial\zeta^2}-\frac{\Lambda}{P}\frac{\partial H}{\partial\theta}=-\frac{\Lambda}{P}\cdot P_0^2\cdot\sin\theta$$

Proceeding as earlier we get:

$$\frac{\partial^2 G}{\partial\zeta^2}-\left(1+i\frac{\Lambda}{P}\right)G=-\frac{\Lambda}{P}P_0^2$$

with the particular solution:

(13) $$G=P_0^2\frac{\frac{\Lambda}{P}}{\left(1+\left(\frac{\Lambda}{P}\right)^2\right)}\left(1-i\frac{\Lambda}{P}\right)$$

The solution to the homogenous equation is:

(14) $$G=g_1(\zeta)+i\,g_2(\zeta)$$

where:

(15) $$g_1(\zeta)=(a\cdot\cosh\alpha\zeta+c\sinh\alpha\zeta)\cos\beta\zeta-(d\cosh\alpha\zeta+b\sinh\alpha\zeta)\sin\beta\zeta$$

(16) $$g_2(\zeta)=(b\cosh\alpha\zeta+d\sin\alpha\zeta)\cos\beta\zeta+(c\cosh\alpha\zeta+a\sinh\alpha\zeta)\sin\beta\zeta$$

(17) $$\alpha^2=\frac{1+\sqrt{1+\left(\frac{\Lambda}{P}\right)^2}}{2}\qquad\beta^2=\frac{-1+\sqrt{1+\left(\frac{\Lambda}{P}\right)^2}}{2}$$

(see Ref. 1)

Thus the complete solution is:

$$H = P_0^2 \frac{\phi}{1+(\phi)^2}(\sin\theta - \phi\cos\theta) + g_1(\zeta)\sin\theta + g_2(\zeta)\cos\theta \tag{18}$$

<u>Boundary Condition at $\zeta = \xi : H = 0$</u>

From the boundary condition we get:

$$g_1(\xi) = \frac{-\phi}{1+(\phi)^2}$$

$$g_2(\xi) = \frac{(\phi)^2}{1+(\phi)^2}$$

which gives:

$$\begin{aligned} c &= c_0 - \varkappa_1 a - \varkappa_2 b \\ d &= d_0 + \varkappa_2 a - \varkappa_1 b \end{aligned} \tag{19}$$

where

$$c_0 = \frac{2(\phi)}{1+(\phi)^2} \; \frac{\phi\cosh\alpha\xi\cdot\sin\beta\xi - \sinh\alpha\xi\cdot\cos\beta\xi}{\cosh 2\alpha\xi - \cos 2\beta\xi} \tag{20}$$

$$d_0 = \frac{2(\phi)}{1+(\phi)^2} \; \frac{\phi\sinh\alpha\xi\cdot\cos\beta\xi + \cosh\alpha\xi\cdot\sin\beta\xi}{\cosh 2\alpha\xi - \cos 2\beta\xi} \tag{21}$$

$$\varkappa_1 = \frac{\sinh 2\alpha\xi}{\cosh 2\alpha\xi - \cos 2\beta\xi} \tag{22}$$

$$\varkappa_2 = \frac{\sin 2\beta\xi}{\cosh 2\alpha\xi - \cos 2\beta\xi} \tag{23}$$

Boundary Condition at $\zeta = 0$, eq. (10)

Introducing eq. (18) into eq. (10) and setting $\zeta = 0$ gives:

$$(24)\quad a = \frac{\left[\frac{\Lambda^2}{2\rho^2} + \alpha\chi_1 + \beta\chi_2\right]\left[-\lambda + \alpha c_0 - \beta d_0\right] + \left[\beta\chi_1 - \alpha\chi_2\right]\left[\frac{\Lambda}{\rho}\lambda + \beta c_0 + \alpha d_0 - \frac{3\Lambda^2}{\rho}\right]}{\left[\frac{\Lambda^2}{2\rho^2} + \alpha\chi_1 + \beta\chi_2\right]^2 + \left[\beta\chi_1 - \alpha\chi_2\right]^2}$$

$$(25)\quad b = \frac{\left[\frac{\Lambda^2}{2\rho^2} + \alpha\chi_1 + \beta\chi_2\right]\left[\frac{\Lambda}{\rho}\lambda + \beta c_0 + \alpha d_0 - \frac{3\Lambda^2}{\rho}\right] - \left[\beta\chi_1 - \alpha\chi_2\right]\left[-\lambda + \alpha c_0 - \beta d_0\right]}{\left[\frac{\Lambda^2}{2\rho^2} + \alpha\chi_1 + \beta\chi_2\right]^2 + \left[\beta\chi_1 - \alpha\chi_2\right]^2}$$

where

$$(26)\qquad \lambda = \frac{\frac{\Lambda}{\rho}}{1 + \left(\frac{\Lambda}{\rho}\right)^2}\left[\frac{\Lambda^2}{2\rho^2}(1 + \rho^2\xi) + \rho^2\right]$$

Together with eq. (19) this defines the arbitrary constants in eq. (15) and (16), thus completing the solution of Reynolds equation.

Force Calculation

The radial and tangential force components are:

$$F_r = -2P_a R^2 \int_0^{\xi}\int_0^{2\pi} P\cos\theta \, d\theta \, d\zeta = -2P_a R^2 \varepsilon \int_0^{\xi}\int_0^{2\pi} \frac{H}{P_0}\cos\theta \, d\theta \, d\zeta$$

$$F_t = 2P_a R^2 \int_0^{\xi}\int_0^{2\pi} P\sin\theta \, d\theta \, d\zeta = 2P_a R^2 \varepsilon \int_0^{\xi}\int_0^{2\pi} \frac{H}{P_0}\sin\theta \, d\theta \, d\zeta$$

where H is given by eq. (18) and P_o by eq. (4). Then:

$$(27)\quad \frac{4F_r}{\pi D L P_a \varepsilon} = \frac{\left(\frac{\Lambda}{\rho}\right)^2}{1 + \left(\frac{\Lambda}{\rho}\right)^2}\,\frac{4\left[(1+\rho^2\xi)^{3/2} - 1\right]}{3\rho^2\xi} - \frac{2}{\xi}\int_0^{\xi}\frac{g_2(\zeta)\,d\zeta}{\sqrt{1 + \rho^2(\xi - \zeta)}}$$

$$(28)\quad \frac{4F_t}{\pi D L P_a \varepsilon} = \frac{\frac{\Lambda}{\rho}}{1 + \left(\frac{\Lambda}{\rho}\right)^2}\,\frac{4\left[(1+\rho^2\xi)^{3/2} - 1\right]}{3\rho^2\xi} + \frac{2}{\xi}\int_0^{\xi}\frac{g_1(\zeta)\,d\zeta}{\sqrt{1 + \rho^2(\xi - \zeta)}}$$

The integrals are evaluated by numerical integration.

The attitude angle φ is given by:

$$\varphi = \mathrm{Arctan}\left(\frac{F_t}{F_r}\right) \tag{29}$$

The total bearing load is given by:

$$W = \sqrt{F_r^2 + F_t^2} \tag{30}$$

Curves of the attitude angle φ and the dimensionless bearing load $\frac{4W}{\pi D L P_a \varepsilon}$ are given in Fig. 1-24 for $\frac{\Lambda}{\bar{P}} = \Lambda$ (i.e. $P = P_a$) and for $\frac{\Lambda}{\bar{P}} = \frac{\Lambda}{P_{0,av}}$ where $P_{o,av}$ is the average value of P_o and consequently of P. The average value of P_o is:

$$P_{0,av} = \frac{2\left[(1+\Lambda^2\xi)^{3/2} - 1\right]}{3\Lambda^2\xi} \tag{31}$$

<u>No Feeding, $\Lambda_t = 0$</u>

In this case we get:

$$a = \frac{x_1 c_0 - x_2 d_0}{x_1^2 + x_2^2} = \frac{\Lambda}{1+\Lambda^2} \cdot \frac{\Lambda \sinh\alpha\xi \cdot \sin\beta\xi - \cosh\alpha\xi \cdot \cos\beta\xi}{\sinh^2\alpha\xi + \cos^2\beta\xi}$$

$$b = \frac{x_2 c_0 + x_1 d_0}{x_1^2 + x_2^2} = \frac{\Lambda}{1+\Lambda^2} \frac{\Lambda \cosh\alpha\xi \cdot \cos\beta\xi + \sinh\alpha\xi \cdot \sin\beta\xi}{\sinh^2\alpha\xi + \cos^2\beta\xi}$$

and therefore $c = d = 0$. Comparing these results to Ref. 1 it is seen that $a = \frac{\Lambda}{1+\Lambda^2} B$ and $b = \frac{\Lambda}{1+\Lambda^2} A$. Thus the present analysis reduces to Ref. 1 (the 1st order pertubation solution) for the case of $\Lambda_t = 0$.

<u>No Rotation, $\Lambda = 0$</u>

In this case $a = c = 0$ and

$$b = -\frac{\frac{\xi}{2}\rho^2 \cdot \sinh\xi}{\cosh\xi + \frac{2}{\xi\rho^2}\sinh\xi}$$

$$d = -\frac{\cosh\xi}{\sinh\xi}\, b = \frac{\frac{\xi}{2}\rho^2 \cosh\xi}{\cosh\xi + \frac{2}{\xi\rho^2}\sinh\xi}$$

Therefore the tangential force $F_t = 0$, i.e., $\varphi = 0$, and the bearing load becomes equal to F_r. We get:

$$\frac{4W}{\pi D L P_a \varepsilon} = \frac{3\rho^2}{\xi\left[\cosh\xi + \frac{2}{\xi\rho^2}\sinh\xi\right]}\int_0^{\xi}\frac{\sinh(\xi-\zeta)}{\sqrt{1+\rho^2(\xi-\zeta)}}\,d\zeta$$

or

$$(32)\qquad \frac{4W}{\pi D L P_a \varepsilon} = \frac{3\rho}{\xi\left[\cosh\xi + \frac{2}{\xi\rho^2}\sinh\xi\right]}\left\{e^{-\frac{1}{\rho^2}}\left[\psi\left(\sqrt{\tfrac{1}{\rho^2}+\xi}\right)-\psi\left(\tfrac{1}{\rho}\right)\right] - \frac{\sqrt{\pi}}{2}e^{\frac{1}{\rho^2}}\left[\varphi\left(\sqrt{\tfrac{1}{\rho^2}+\xi}\right)-\varphi\left(\tfrac{1}{\rho}\right)\right]\right\}$$

where

$$\psi(x) = \int_0^x e^{t^2}dt \qquad\qquad \varphi(x) = \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

This may be compared to Ref. 3 by means of the following equations:

$$\bar{\xi} = \frac{\xi \Lambda_t^2}{\rho^2}$$

$$U = \sqrt{\frac{\xi(1+\bar{\xi})}{\bar{\xi}\,(V^2-1)}} = \frac{1}{\rho}$$

Then eq. (32) may be rewritten:

$$\frac{4W}{\pi D L P_a \varepsilon} = \frac{3}{U}\left\{\frac{e^{-U^2}\left[\psi\left(\sqrt{U^2+\xi}\right)-\psi(U)\right] - \frac{\sqrt{\pi}}{2}\left[\varphi\left(\sqrt{U^2+\xi}\right)-\varphi(U)\right]}{\xi\cosh\xi + \frac{1}{2}\,\bar{\xi}\,\sinh\xi}\right\}$$

which is the same as eq. (19A), page 259, of Ref. 3.

<u>A Check on the Assumption $\hat{P}$ = constant</u>

Since $P_0 = \sqrt{1+\rho^2(\xi-\zeta)}$ is reasonably linear we shall substitute linear

expressions for $\overset{\Delta}{P}_o$ and ΛP_o into eq. (5) and thereby obtain an indication of the validity of the previous assumption $\overset{\Delta}{P}$ = constant.

<u>Homogenous Equation</u>

In eq. (12) we introduce the approximation

(33) $$\overset{\Delta}{P}_0 = \Lambda\left[\frac{1}{\sqrt{1+\mu^2\xi}} + \frac{1}{\xi}\left(1-\frac{1}{\sqrt{1+\mu^2\xi}}\right)\zeta\right] = \Lambda\varkappa + \frac{\zeta}{q}$$

where

(34) $$\varkappa = \frac{1}{\sqrt{1+\mu^2\xi}}$$

(35) $$q = \frac{\xi}{\Lambda(1-\varkappa)}$$

Furthermore set:

(35a) $$z = 1 + i\overset{\Delta}{P}_0 = 1 + i\left(\Lambda\varkappa + \frac{\zeta}{q}\right) \qquad \begin{matrix}\Lambda \neq 0\\ \Lambda_t \neq 0\end{matrix}$$

Then the homogenous part of eq. (12) becomes:

(36) $$\frac{d^2G}{dz^2} + q^2 z G = 0$$

with the solution:

$$G = \sqrt{z}\left[A\cdot J_{\frac{1}{3}}\left(\tfrac{2}{3}qz^{3/2}\right) + B\cdot J_{-\frac{1}{3}}\left(\tfrac{2}{3}qz^{3/2}\right)\right]$$

Since Z is a complex variable this expression is rather cumbersome to evaluate. Instead the following approximation shall be used:

(37) $$G = z^{-1/4}\left[A\cdot\cos\left(\tfrac{2}{3}qz^{3/2}\right) + B\cdot\sin\left(\tfrac{2}{3}qz^{3/2}\right)\right]$$

To evaluate this approximation insert eq. (37) into eq. (36) to get:

(38) $$\left(\frac{5}{16}z^{-2} - q^2 z\right)G + q^2 z G \cong 0$$

Comparing the real and the imaginary components:

(39) real part: $$q^2 \cong q^2\left[1 - \frac{5}{16}\,\frac{1-(\Lambda x+\frac{\xi}{q})^2}{q^2[1+(\Lambda x+\frac{\xi}{q})^2]^2}\right]$$

(40) imaginary part: $$(\Lambda x+\tfrac{\xi}{q}) \cong (\Lambda x+\tfrac{\xi}{q})\left[1+\frac{5}{8}\,\frac{1}{q^2[1+(\Lambda x+\frac{\xi}{q})^2]^2}\right]$$

Thus for eq. (37) to be a good approximation the remainders inside the above brackets must be small compared to unity. The maximum values of the remainders are:

(41) Max. remainder for real part: $$\frac{5}{16}\,\frac{1-\Lambda^2x^2}{q^2[1+\Lambda^2x^2]^2}$$

(42) Max. remainder for imaginary part: $$\frac{5}{8}\,\frac{1}{q^2[1+\Lambda^2x^2]^2}$$

These equations have been plotted in Fig. 13-20 for a few representative values of Λ_t, V and Λ and thus indicates the regions where the results of the analysis is valid.

Returning to eq. (37) we set:

$$\tfrac{2}{3}\,q z^{3/2} = \alpha + i\beta$$

and

$$z^{-1/4} = \gamma + i\delta$$

From eq. (35a) we get:

(43) $$\alpha^2 = \frac{4q^2}{9}\left[\frac{-\left(3(\frac{\Lambda}{k})^2-1\right)+\sqrt{\left(1+(\frac{\Lambda}{k})^2\right)^3}}{2}\right]$$

(44) $$\beta^2 = \frac{4q^2}{9}\left[\frac{\left(3(\frac{\Lambda}{k})^2-1\right)+\sqrt{\left(1+(\frac{\Lambda}{k})^2\right)^3}}{2}\right]$$

(45) $$\gamma^2 = \frac{1-\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{1+\left(\frac{A}{p_0}\right)^2}}\right)}}{2\left[1+\left(\frac{A}{p_0}\right)^2\right]^{1/4}}$$

(46) $$\delta^2 = \frac{1+\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{1+\left(\frac{A}{p_0}\right)^2}}\right)}}{2\left[1+\left(\frac{A}{p_0}\right)^2\right]^{1/4}}$$

where

(47) $$\frac{A}{p_0} = \Lambda x + \frac{\zeta}{q}$$

Introducing the above equations into eq. (37), setting A = a+ib and B = c+id and using eq. (12a) yields:

(48) $$q_1(\zeta) = a[\gamma\cos\alpha\cosh\beta + \delta\sin\alpha\sinh\beta] - b[\delta\cos\alpha\cosh\beta - \gamma\sin\alpha\sinh\beta] + c[\gamma\sin\alpha\cosh\beta - \delta\cos\alpha\sinh\beta] - d[\delta\sin\alpha\cosh\beta + \gamma\cos\alpha\sinh\beta]$$

(49) $$q_2(\zeta) = a[\delta\cos\alpha\cosh\beta - \gamma\sin\alpha\sinh\beta] + b[\gamma\cos\alpha\cosh\beta + \delta\sin\alpha\sinh\beta] + c[\delta\sin\alpha\cosh\beta + \gamma\cos\alpha\sinh\beta] + d[\gamma\sin\alpha\cosh\beta - \delta\cos\alpha\sinh\beta]$$

Complete Equation

A particular solution to the complete inhomogenous eq. (12) can be found either by using the substitution given by eq. (35a) to obtain a series solution or by proceeding with the homogenous solution by means of variation of parameters. Neither method yields very practical results. Instead we return to eq. (5) and make the following approximations:

$$\frac{\Lambda}{P_0} = \frac{1}{\sqrt{1+\rho^2\xi} + \frac{1}{\xi}\left(1-\sqrt{1+\rho^2\xi}\right)\zeta} = \frac{\Lambda^2 \varkappa q}{\Lambda q - \zeta}$$

$$\Lambda P_0 = \frac{\Lambda}{\varkappa} \cdot \frac{\sinh(\xi-\zeta) + \varkappa \sinh\zeta}{\sinh\xi}$$

Note that for $\xi < 1$ these two approximations for P_o are nearly identical. Then a particular solution of eq. (5) is:

$$H = \frac{\Lambda/2\varkappa}{\left(1+\frac{\Lambda^2\varkappa^2 q^2}{4}\right)\sinh\xi}(\zeta - \Lambda q)\left[\left(\cosh(\xi-\zeta) - \varkappa\cosh\zeta\right)\sin\theta + \frac{\Lambda^2\varkappa q}{\xi}\left(\sinh(\xi-\zeta) + \varkappa\sinh\zeta\right)\right] \tag{50}$$

Combining eq. (11), eq. (48), eq. (49), and eq. (50) givės the complete solution to eq. (5).

<u>Boundary Condition at $\zeta = \xi$: $H = 0$</u>

The boundary condition becomes:

$$g_1(\xi) = \frac{\frac{\Lambda^2\varkappa q}{\xi}}{1+\frac{\Lambda^2\varkappa^2 q^2}{4}} \frac{1-\varkappa\cosh\xi}{\varkappa\sinh\xi} = a[\gamma_1\cos\alpha_1\cosh\beta_1 + \delta_1\sin\alpha_1\sinh\beta_1] - b[\delta_1\cos\alpha_1\cosh\beta_1 - \gamma_1\sin\alpha_1\sinh\beta_1]$$
$$+ c[\gamma_1\sin\alpha_1\cosh\beta_1 - \delta_1\cos\alpha_1\sinh\beta_1] - d[\delta_1\sin\alpha_1\cosh\beta_1 + \gamma_1\cos\alpha_1\sinh\beta_1] \tag{51}$$

$$g_2(\xi) = \frac{\frac{\Lambda^2\varkappa^2 q^2}{\xi}}{1+\frac{\Lambda^2\varkappa^2 q^2}{4}} = a[\delta_1\cos\alpha_1\cosh\beta_1 - \gamma_1\sin\alpha_1\sinh\beta_1] + b[\gamma_1\cos\alpha_1\cosh\beta_1 + \delta_1\sin\alpha_1\sinh\beta_1]$$
$$+ c[\delta_1\sin\alpha_1\cosh\beta_1 + \gamma_1\cos\alpha_1\sinh\beta_1] + d[\gamma_1\sin\alpha_1\cosh\beta_1 - \delta_1\cos\alpha_1\sinh\beta_1] \tag{52}$$

where α_1, β_1, γ_1 and δ_1 are derived from eq. (43)-(46) by setting $\zeta = \xi$ (i.e., $P_o = 1$).

Solving eq. (51) and (52) with respect to c and d yields:

$$c = c_0 - \lambda_1 a - \lambda_2 b \tag{53}$$

$$d = d_0 + \lambda_2 a - \lambda_1 b \tag{54}$$

where:

(55) $$C_0 = \frac{2\left[\left(\gamma_1 \cdot q_1(\xi) + \delta_1 \cdot q_2(\xi)\right)\sin\alpha_1 \cosh\beta_1 + \left(\gamma_1 \cdot q_2(\xi) - \delta_1 \cdot q_1(\xi)\right)\cos\alpha_1 \sinh\beta_1\right]}{\left(\gamma_1^2 + \delta_1^2\right)\left(\cosh 2\beta_1 - \cos 2\alpha_1\right)}$$

(56) $$d_0 = \frac{2\left[\left(\gamma_1 \cdot q_2(\xi) - \delta_1 \cdot q_1(\xi)\right)\sin\alpha_1 \cosh\beta_1 - \left(\gamma_1 \cdot q_1(\xi) + \delta_1 \cdot q_2(\xi)\right)\cos\alpha_1 \sinh\beta_1\right]}{\left(\gamma_1^2 + \delta_1^2\right)\left(\cosh 2\beta_1 - \cos 2\alpha_1\right)}$$

(57) $$\lambda_1 = \frac{\sin 2\alpha_1}{\cosh 2\beta_1 - \cos 2\alpha_1}$$

(58) $$\lambda_2 = \frac{\sinh 2\beta_1}{\cosh 2\beta_1 - \cos 2\alpha_1}$$

__Boundary Condition at $\zeta = 0$, see eq. (10)__

Introduce the following constant:

(59) $$\alpha_0 = (\alpha)_{\zeta=0}$$

(60) $$\beta_0 = (\beta)_{\zeta=0}$$

(61) $$\gamma_0 = (\gamma)_{\zeta=0}$$

(62) $$\delta_0 = (\delta)_{\zeta=0}$$

(63) $$\alpha_0' = \left(\frac{d\alpha}{d\zeta}\right)_{\zeta=0} = \frac{1}{3}\,\frac{\xi x}{1-x} \cdot \frac{\left(-2 + \sqrt{1 + (\Lambda x)^2}\right)}{\alpha_0}$$

(64) $$\beta_0' = \left(\frac{d\beta}{d\zeta}\right)_{\zeta=0} = \frac{1}{3}\,\frac{\xi x}{1-x}\,\frac{\left(2 + \sqrt{1 + (\Lambda x)^2}\right)}{\beta_0}$$

(65) $$\gamma_0' = \left(\frac{d\gamma}{d\zeta}\right)_{\zeta=0} = \frac{1}{8}\,\frac{\Lambda x}{4}\left[\frac{1}{2}\left(\frac{1}{2}\left(1 + \frac{1}{\sqrt{1+\Lambda^2 x^2}}\right)\right)^{-\frac{1}{2}} - \sqrt{1+\Lambda^2 x^2}\left(1 - \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1+\Lambda^2 x^2}}\right)}\right)\right] \Big/ \gamma_0\left(1+\Lambda^2 x^2\right)^{3/4}$$

(66) $$\delta_0' = \left(\frac{d\delta}{d\xi}\right)_{\xi=0} = \frac{1}{8}\frac{\Lambda^2 x}{4}\left[-\frac{1}{2}\left(\frac{1}{2}\left(1+\frac{1}{\sqrt{1+\Lambda^2 x^2}}\right)\right)^{-\frac{1}{2}} - \sqrt{1+\Lambda^2 x^2}\left(1+\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{1+\Lambda^2 x^2}}\right)}\right)\right] \Big/ \delta_0 (1+\Lambda^2 x^2)^{3/4}$$

(67) $$\nu = \gamma_0 \cos\alpha_0 \cosh\beta_0 + \delta_0 \sin\alpha_0 \sinh\beta_0$$

(68) $$\rho = \delta_0 \cos\alpha_0 \cosh\beta_0 - \gamma_0 \sin\alpha_0 \sinh\beta_0$$

(69) $$\sigma = \gamma_0 \sin\alpha_0 \cosh\beta_0 - \delta_0 \cos\alpha_0 \sinh\beta_0$$

(70) $$\tau = \delta_0 \sin\alpha_0 \cosh\beta_0 + \gamma_0 \cos\alpha_0 \sinh\beta_0$$

(71) $$\nu' = \gamma_0' \cos\alpha_0 \cosh\beta_0 + \delta_0' \sin\alpha_0 \sinh\beta_0 - \alpha_0'\sigma + \beta_0'\tau$$

(72) $$\rho' = \delta_0' \cos\alpha_0 \cosh\beta_0 - \gamma_0' \sin\alpha_0 \sinh\beta_0 - \alpha_0'\tau - \beta_0'\sigma$$

(73) $$\sigma' = \gamma_0' \sin\alpha_0 \cosh\beta_0 - \delta_0 \cos\alpha_0 \sinh\beta_0 + \alpha_0'\nu - \beta_0'\rho$$

(74) $$\tau' = \delta_0' \sin\alpha_0 \cosh\beta_0 + \gamma_0' \cos\alpha_0 \sinh\beta_0 + \alpha_0'\rho + \beta_0'\nu$$

(75) $$S = \frac{\Lambda^2 x a}{2}$$

(76) $$\psi = \nu - \lambda_1\sigma - \lambda_2\tau$$

(77) $$\eta = \rho + \lambda_2\sigma - \lambda_1\tau$$

(78) $$\psi' = \nu' - \lambda_1\sigma' - \lambda_2\tau'$$

(79) $$\eta' = \rho' + \lambda_2\sigma' - \lambda_1\tau'$$

$$\Omega = \frac{\Lambda/2\kappa}{1+S^2}\left[\frac{\cosh\xi-\kappa}{\sinh\xi}\left(1+\Lambda q\cdot\frac{A_\xi^2}{2\rho^2}\right)+\Lambda q\right] + \left(\sigma' - \frac{A_\xi^2}{2\rho^2}\sigma\right)c_0 - \left(\tau' - \frac{A_\xi^2}{2\rho^2}\tau\right)d_0 \tag{80}$$

$$\Gamma = S\,\frac{\Lambda/2\kappa}{1+S^2}\left[1+\Lambda q\,\frac{A_\xi^2}{2\rho^2} + \Lambda q\,\frac{\cosh\xi-\kappa}{\sinh\xi}\right] + \left(\tau' - \frac{A_\xi^2}{2\rho^2}\tau\right)c_0 + \left(\sigma' - \frac{A_\xi^2}{2\rho^2}\sigma\right)d_0 \tag{81}$$

Then we get from the boundary condition eq. (10):

$$a = \frac{\Omega\left[\frac{A_\xi^2}{2\rho^2}\psi-\psi'\right] + \left(\Gamma-\frac{3\rho^2}{2}\right)\left[\frac{A_\xi^2}{2\rho^2}\eta-\eta'\right]}{\left[\frac{A_\xi^2}{2\rho^2}\psi-\psi'\right]^2+\left[\frac{A_\xi^2}{2\rho^2}\eta-\eta'\right]^2} \tag{82}$$

$$b = \frac{\left(\Gamma-\frac{3\rho^2}{2}\right)\left[\frac{A_\xi^2}{2\rho^2}\psi-\psi'\right] - \Omega\left[\frac{A_\xi^2}{2\rho^2}\eta-\eta'\right]}{\left[\frac{A_\xi^2}{2\rho^2}\psi-\psi'\right]^2+\left[\frac{A_\xi^2}{2\rho^2}\eta-\eta'\right]^2} \tag{83}$$

Force Calculation

The radial and tangential force components are:

$$F_r = -2P_a R^2 \varepsilon \int_0^{\zeta}\int_0^{2\pi} \frac{H}{P_0}\cos\theta\, d\theta\, d\zeta \tag{84}$$

$$F_t = 2P_a R^2 \varepsilon \int_0^{\zeta}\int_0^{2\pi} \frac{H}{P_0}\sin\theta\, d\theta\, d\zeta \tag{85}$$

where:

(86) $$H = \left\{ \frac{\Lambda/2\varkappa}{(1+S^2)\sinh\xi} (\zeta - \Lambda q)(\cosh(\xi-\zeta) - \varkappa\cosh\zeta) + g_1(\zeta) \right\} \sin\theta$$
$$+ \left\{ S \cdot \frac{\Lambda/2\varkappa}{(1+S^2)\sinh\xi} (\zeta - \Lambda q)(\sinh(\xi-\zeta) + \varkappa\sinh\zeta) + g_2(\zeta) \right\} \cos\theta$$

(87) $$P_o = \sqrt{1 + \mu^2(\xi-\zeta)}$$

and $g_1(\zeta)$ and $g_2(\zeta)$ are given by eq. (48) and (49). Then:

(88) $$\frac{4F_r}{\pi D L R \varepsilon} = -\frac{2}{\xi} \int_0^{\xi} \frac{S \cdot \frac{\Lambda/2\varkappa}{(1+S^2)\sinh\xi} (\zeta - \Lambda q)(\sinh(\xi-\zeta) + \varkappa\sinh\zeta) + g_2(\zeta)}{\sqrt{1+\mu^2(\xi-\zeta)}} \, d\zeta$$

(89) $$\frac{4F_t}{\pi D L R \varepsilon} = \frac{2}{\xi} \int_0^{\xi} \frac{\frac{\Lambda/2\varkappa}{(1+S^2)\sinh\xi} (\zeta - \Lambda q)(\cosh(\xi-\zeta) - \varkappa\cosh\zeta) + g_1(\zeta)}{\sqrt{1+\mu^2(\xi-\zeta)}} \, d\zeta$$

These integrals are evaluated numerically. The total force is:

(90) $$W = \sqrt{F_r^2 + F_t^2}$$

and the attitude angle is determined by:

(91) $$\varphi = \operatorname{Arctan}\left(\frac{F_t}{F_r}\right)$$

FIG. 1

DIMENSIONLESS FORCE

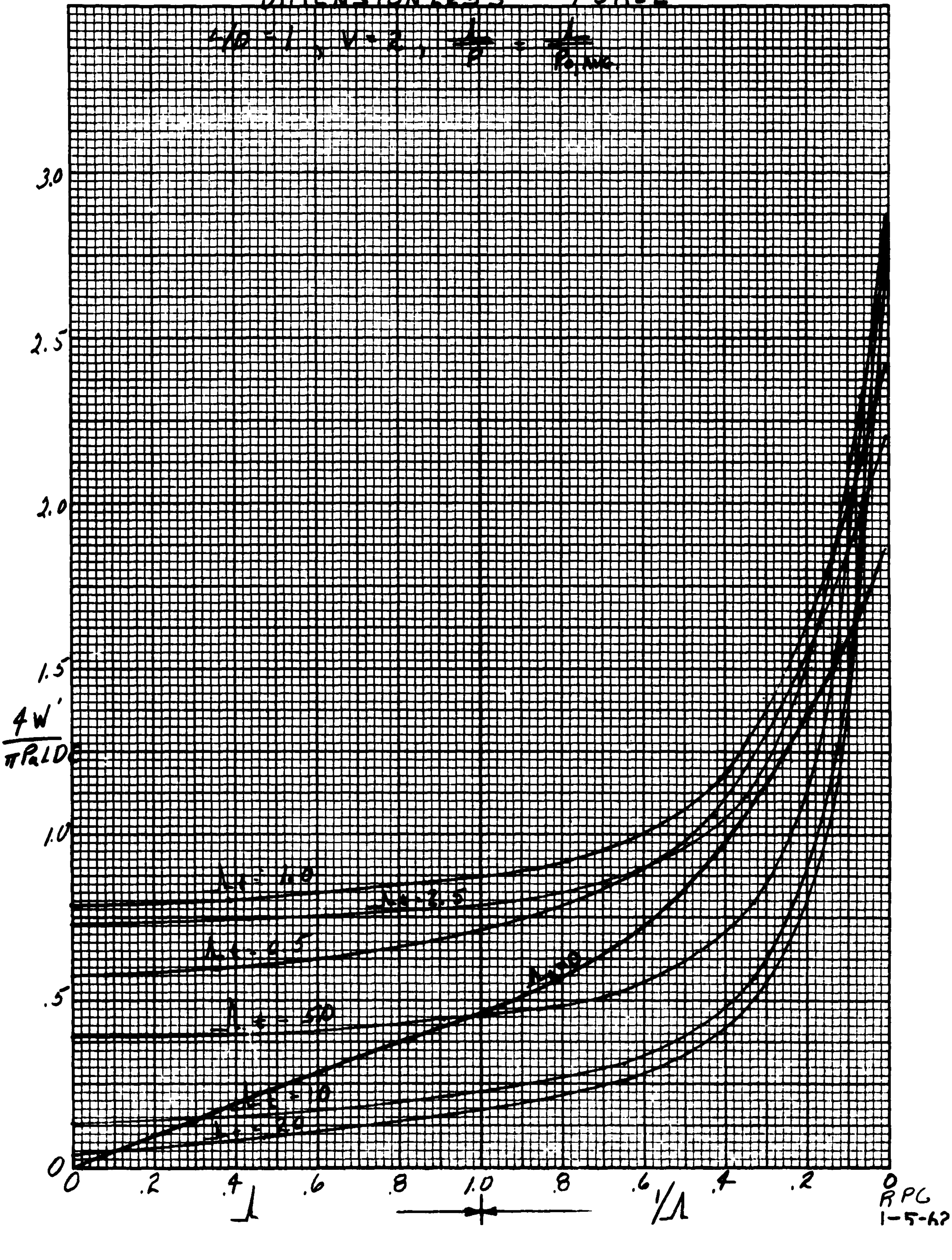

FIG. 2

DIMENSIONLESS FORCE

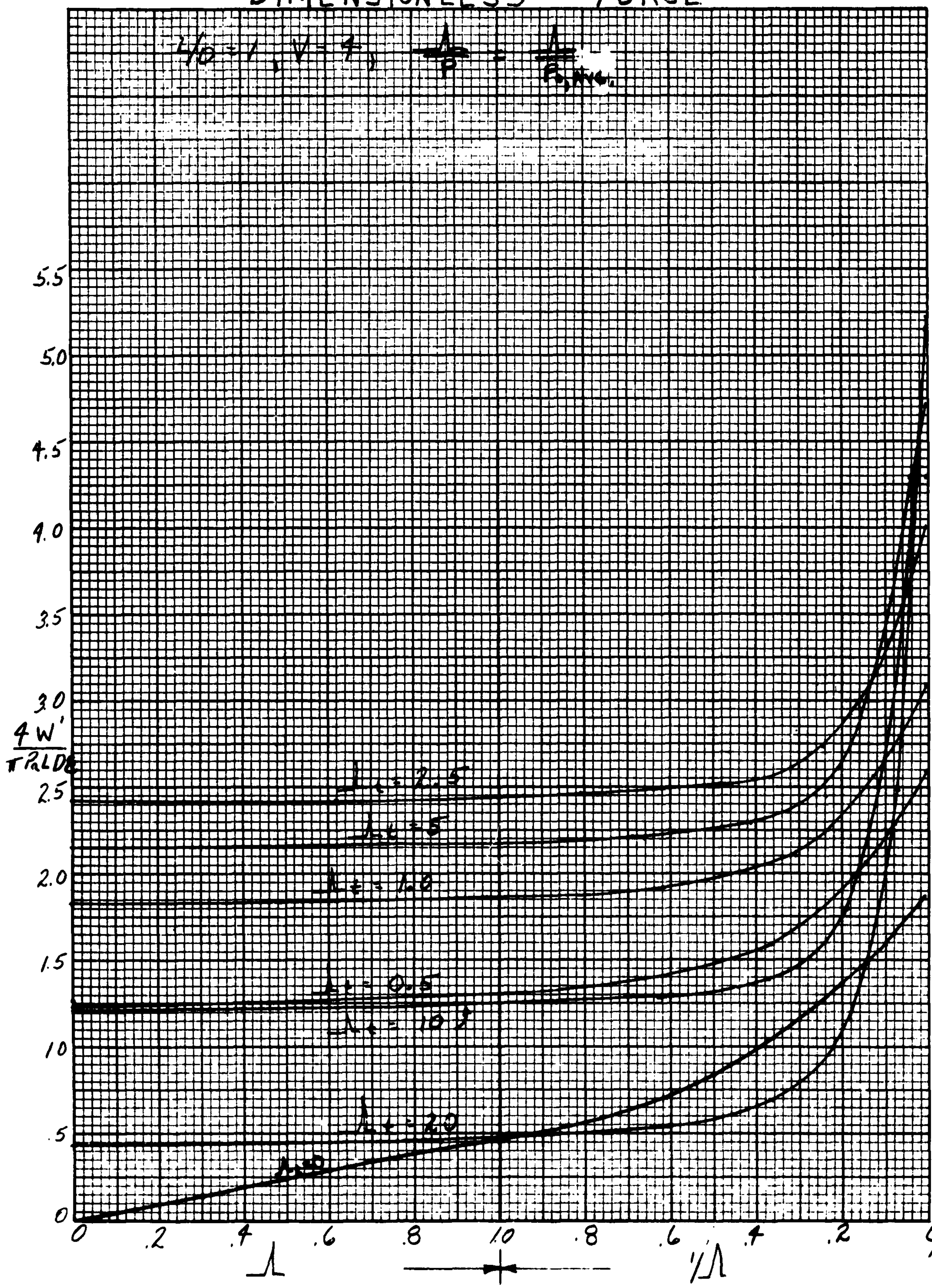

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FIG. 3

DIMENSIONLESS FORCE

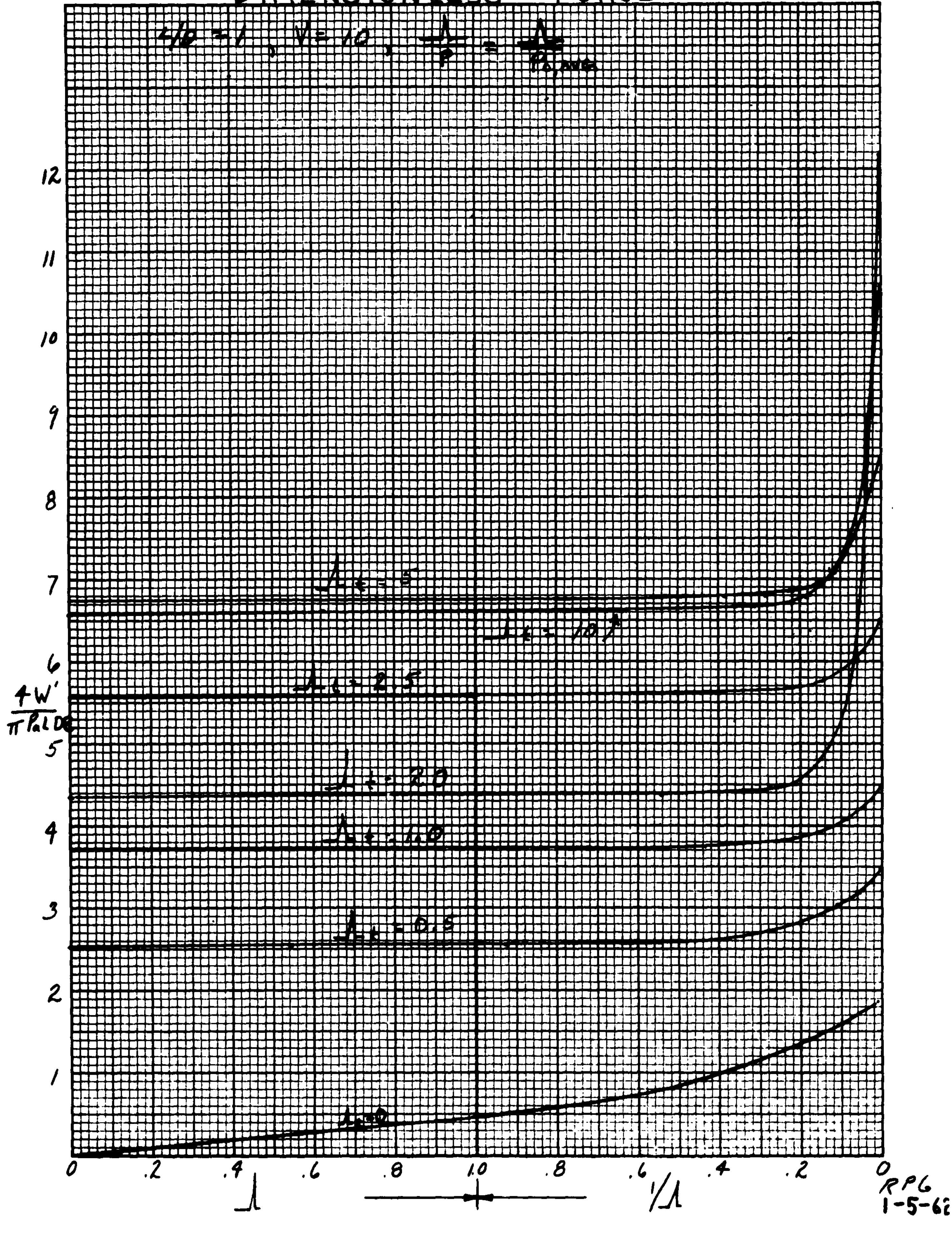

FIG. 4

DIMENSIONLESS FORCE

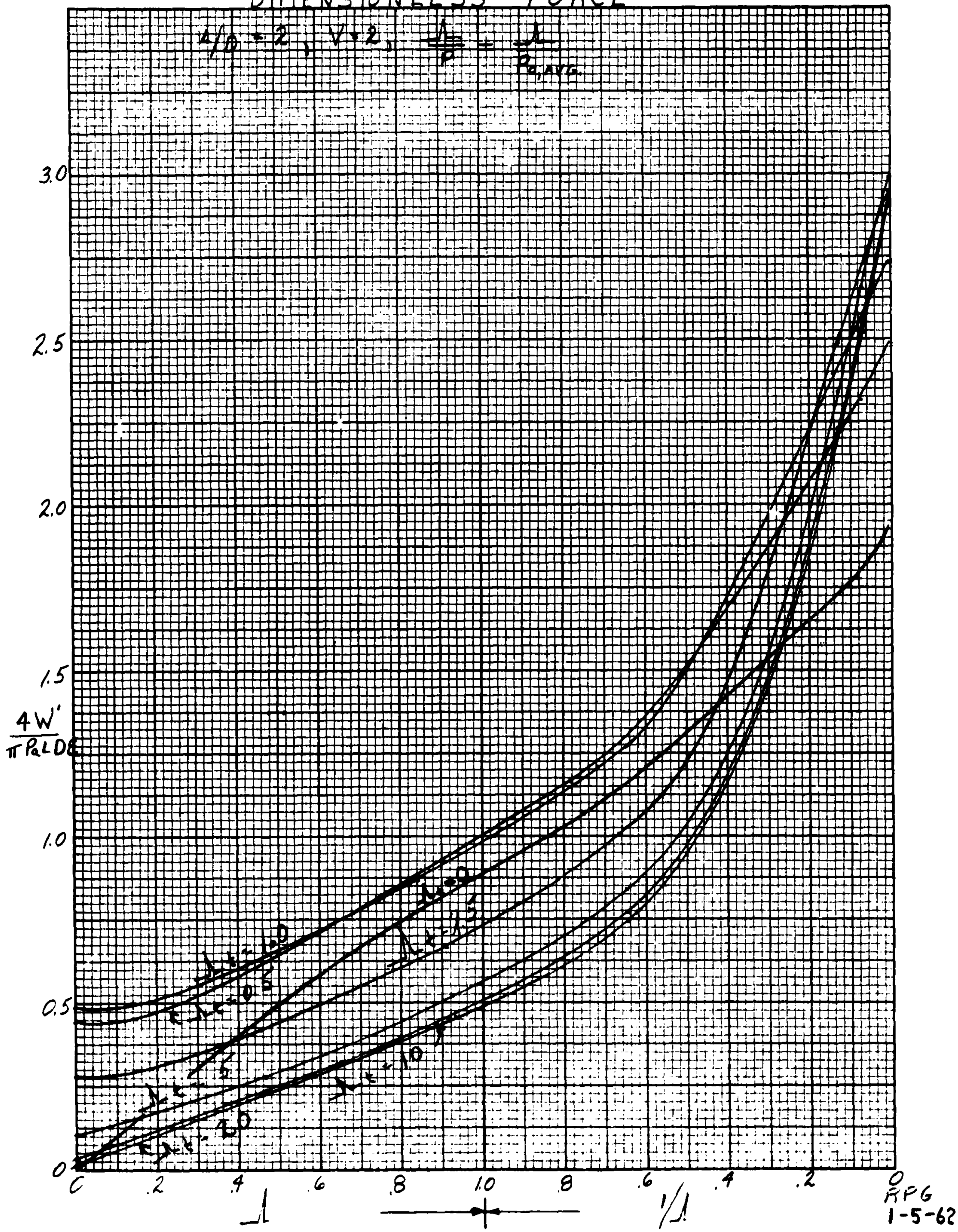

FIG. 5
DIMENSIONLESS FORCE

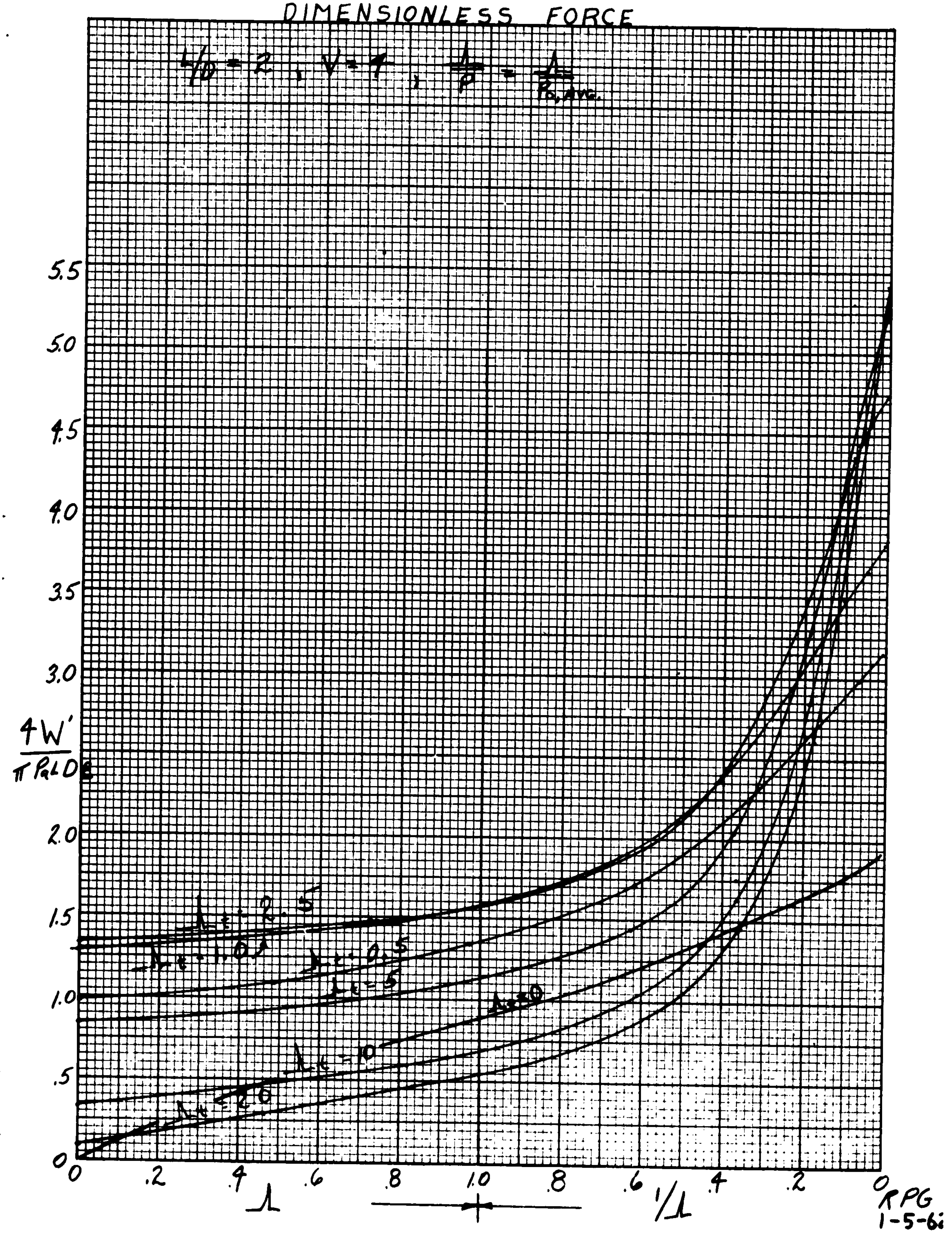

FIG. 6

DIMENSIONLESS FORCE

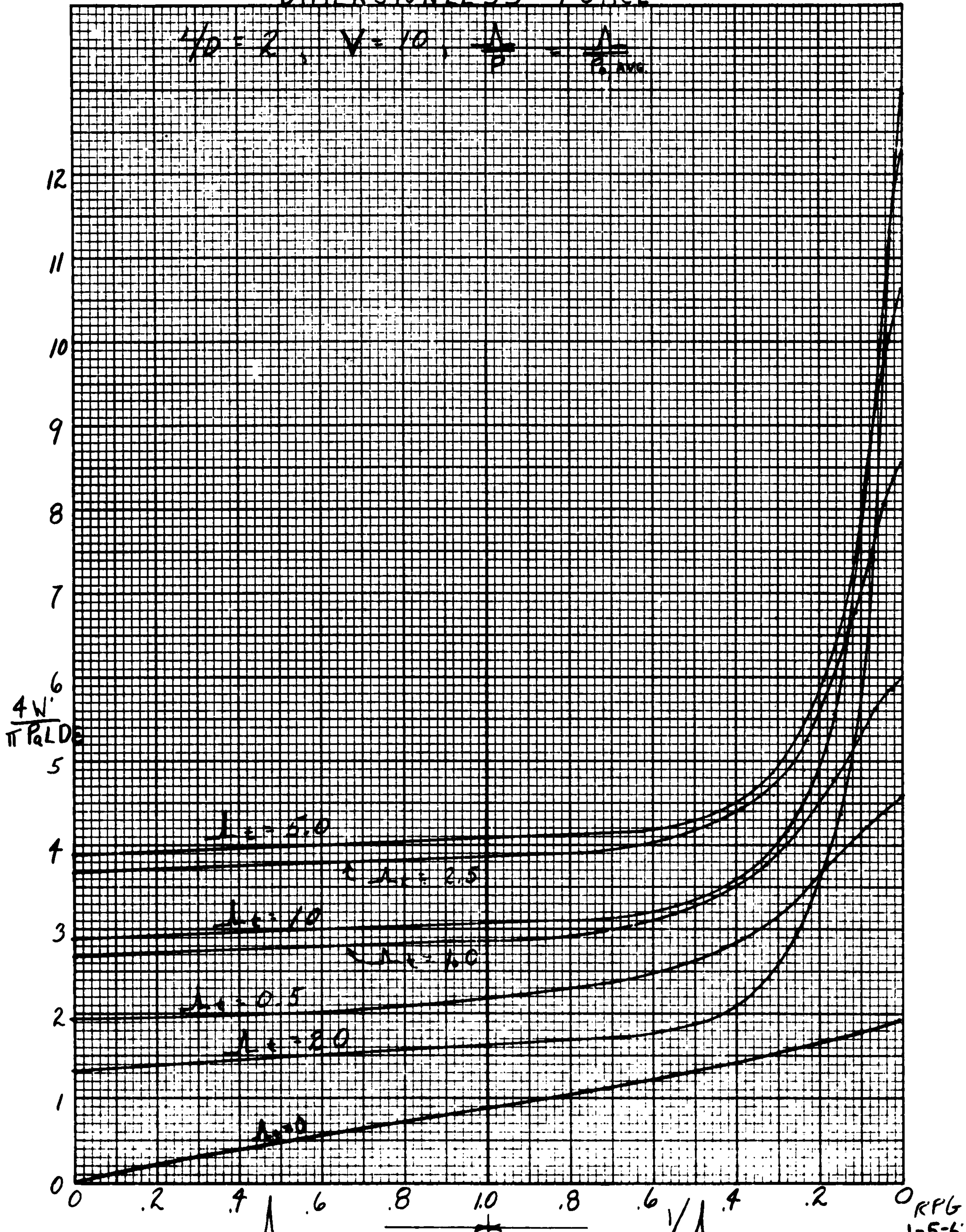

FIG. 7

ATTITUDE ANGLE

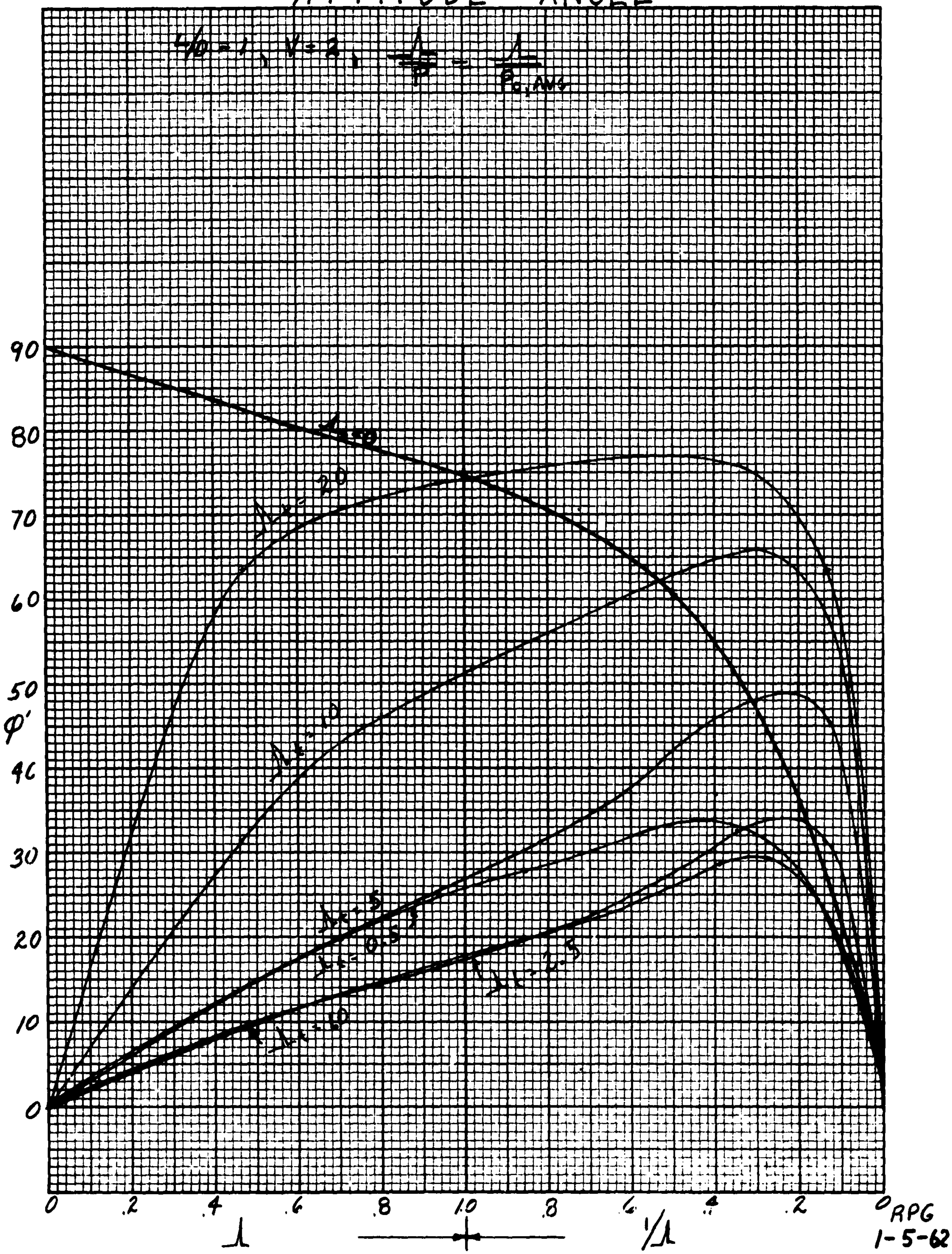

FIG. 8

ATTITUDE ANGLE

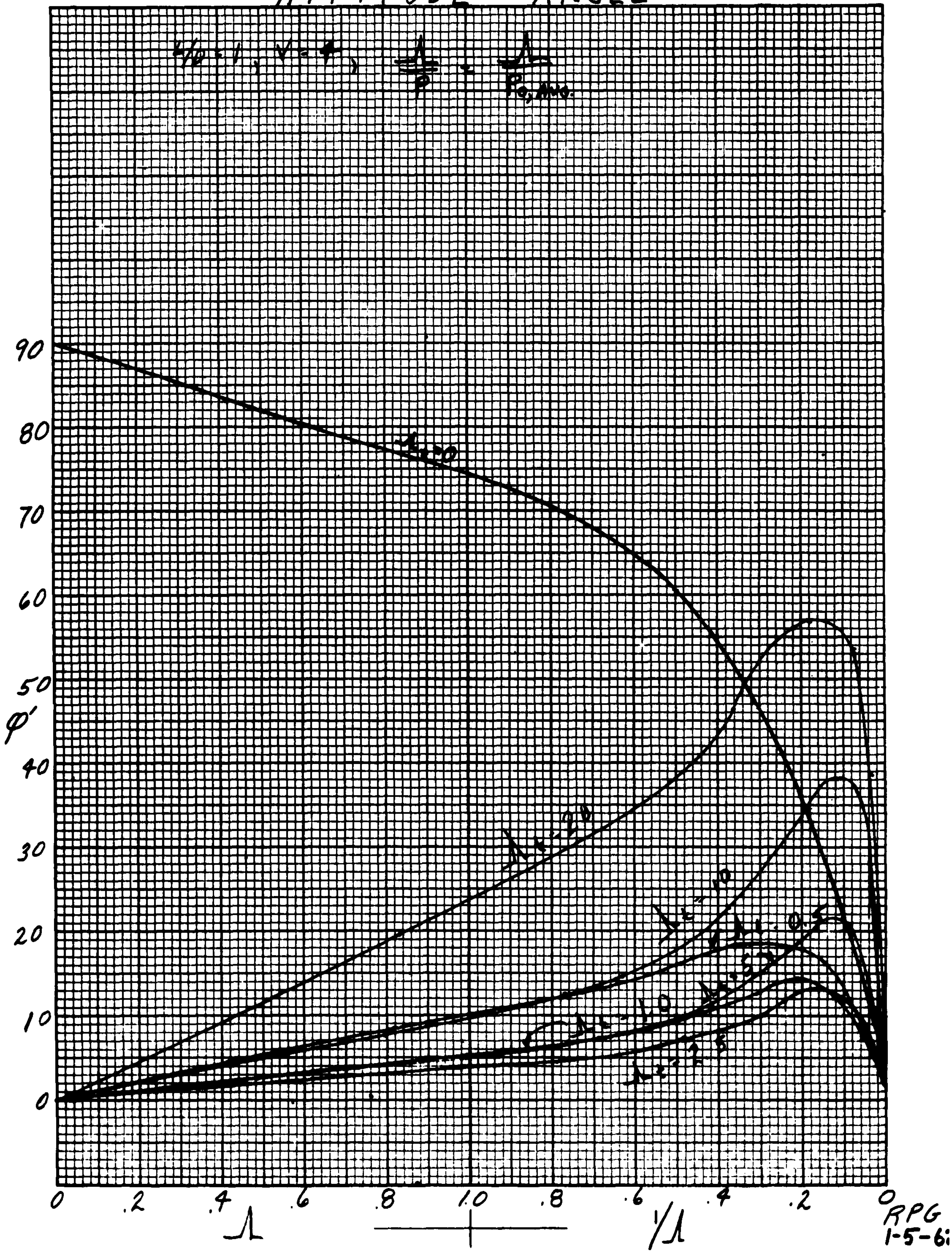

FIG. 9

ATTITUDE ANGLE

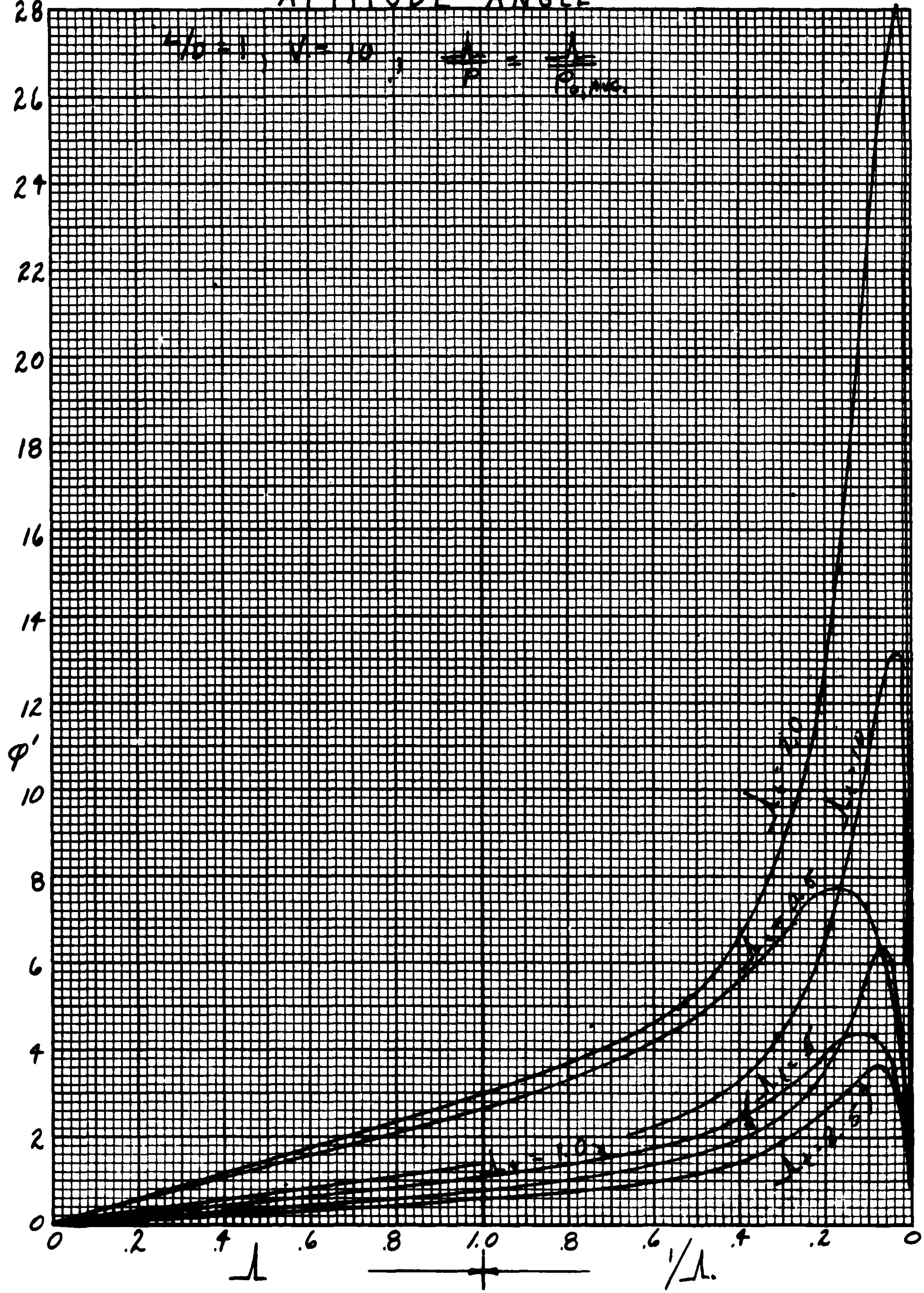

FIG. 10

ATTITUDE ANGLE

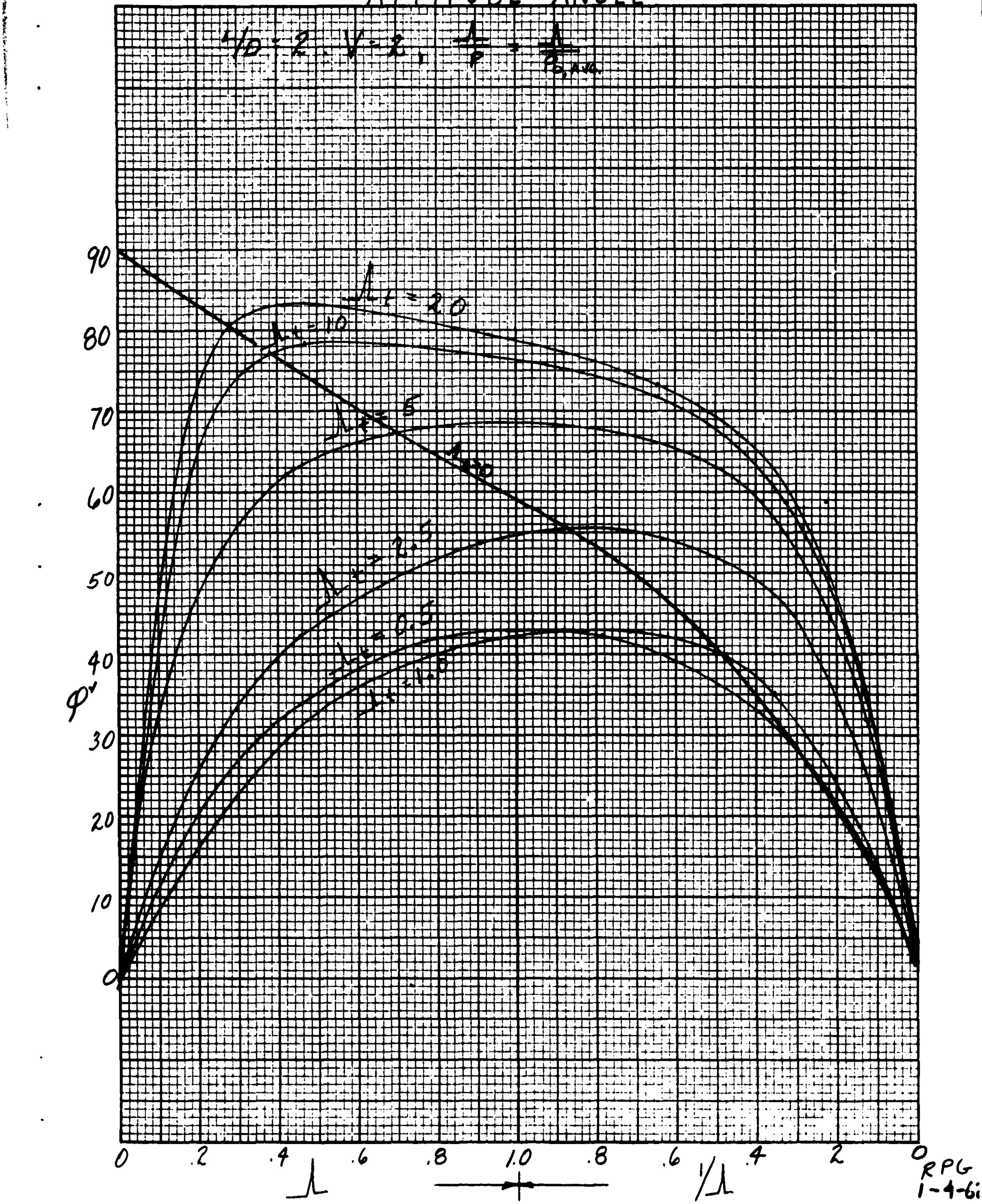

FIG. 11

ATTITUDE ANGLE

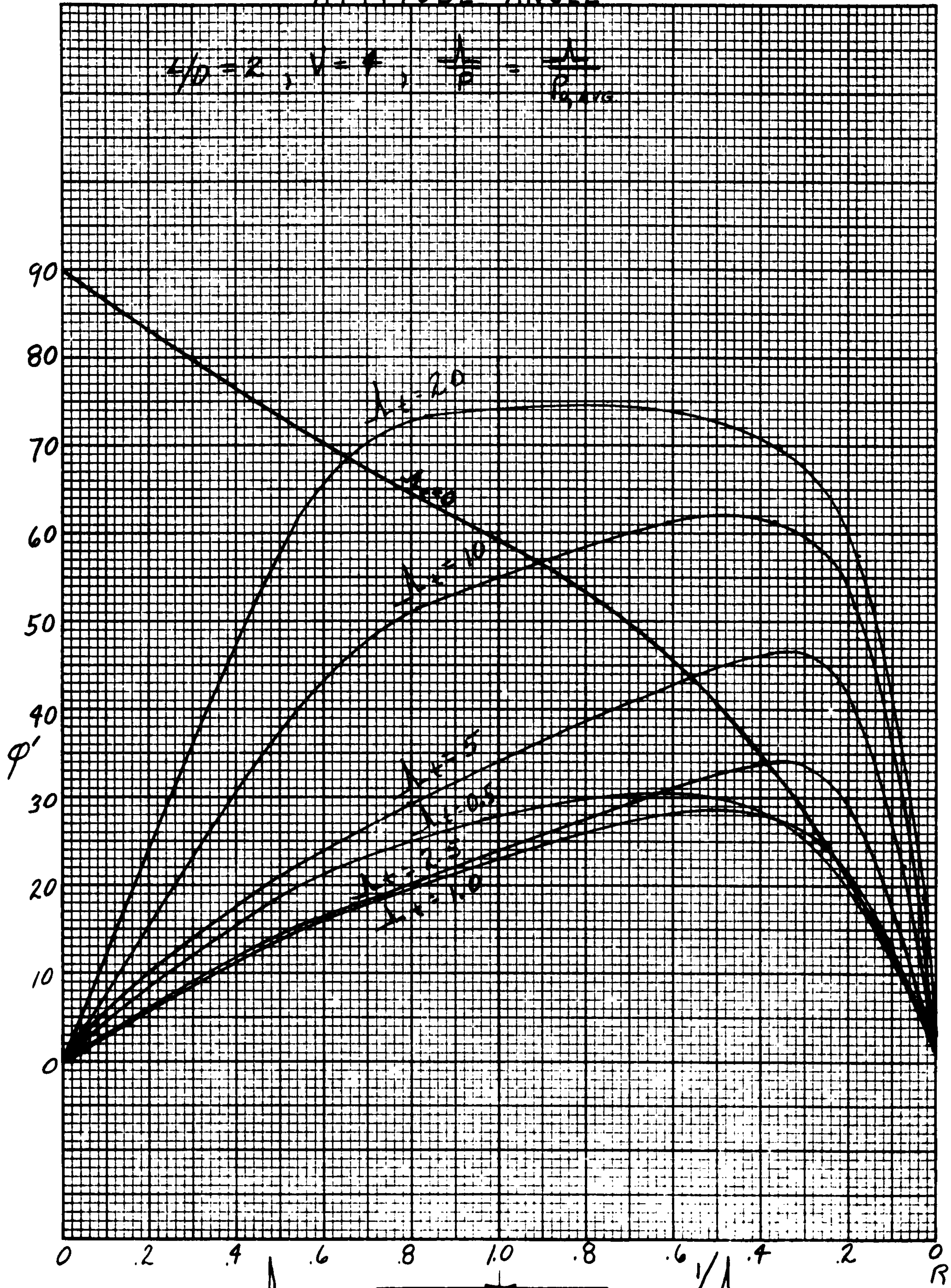

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FIG. 12

ATTITUDE ANGLE

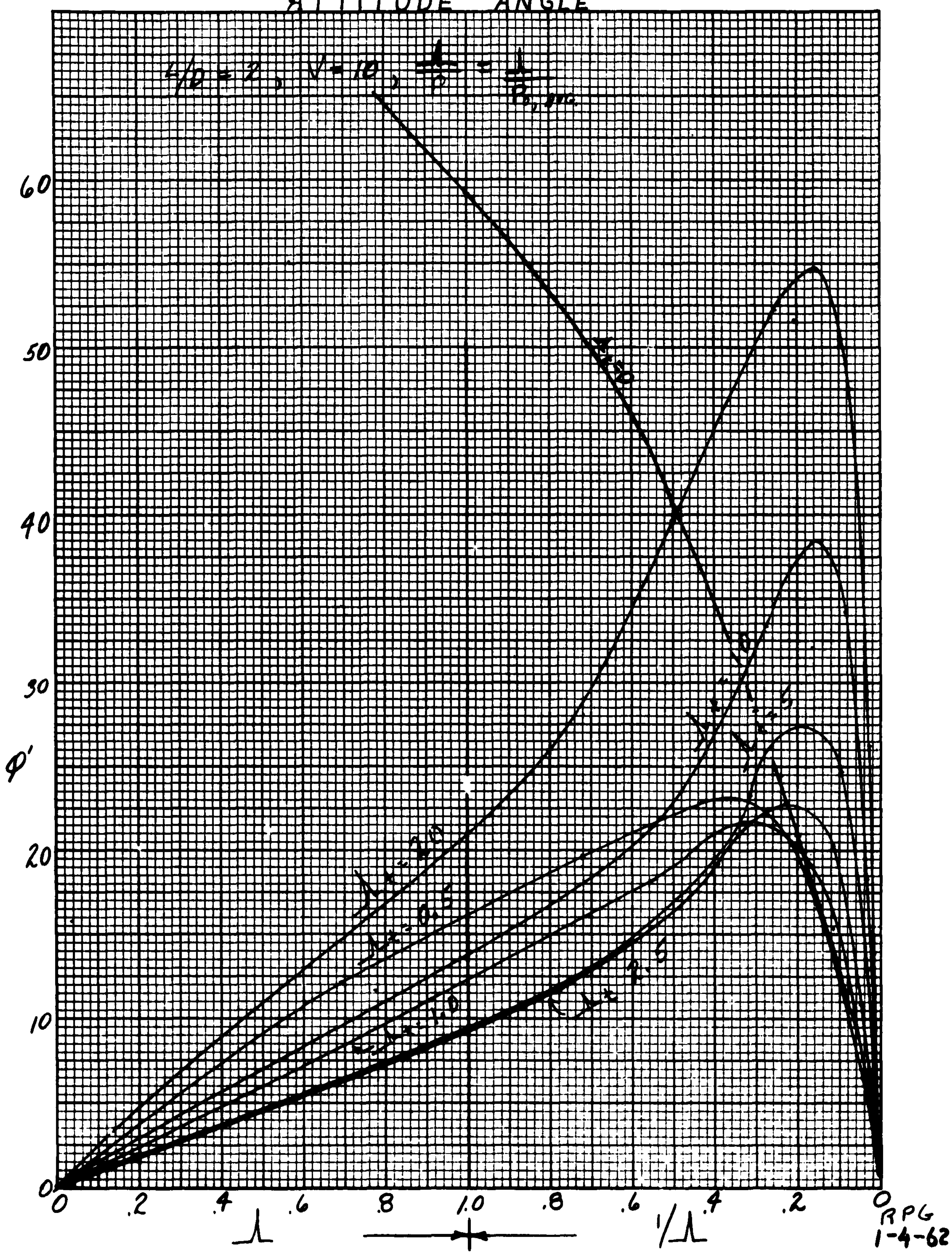


FIG. 13

Error in real part of eq. 38

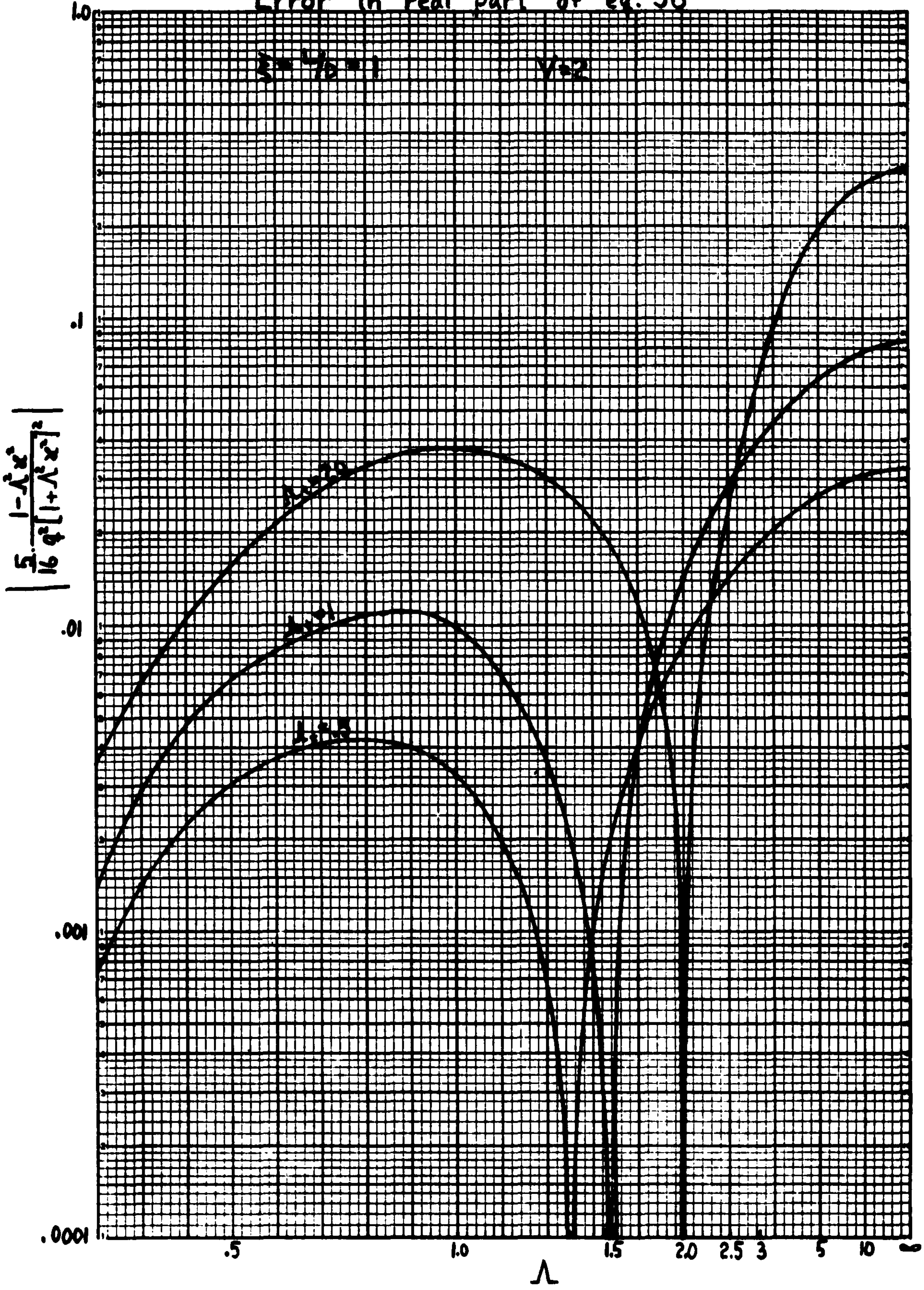
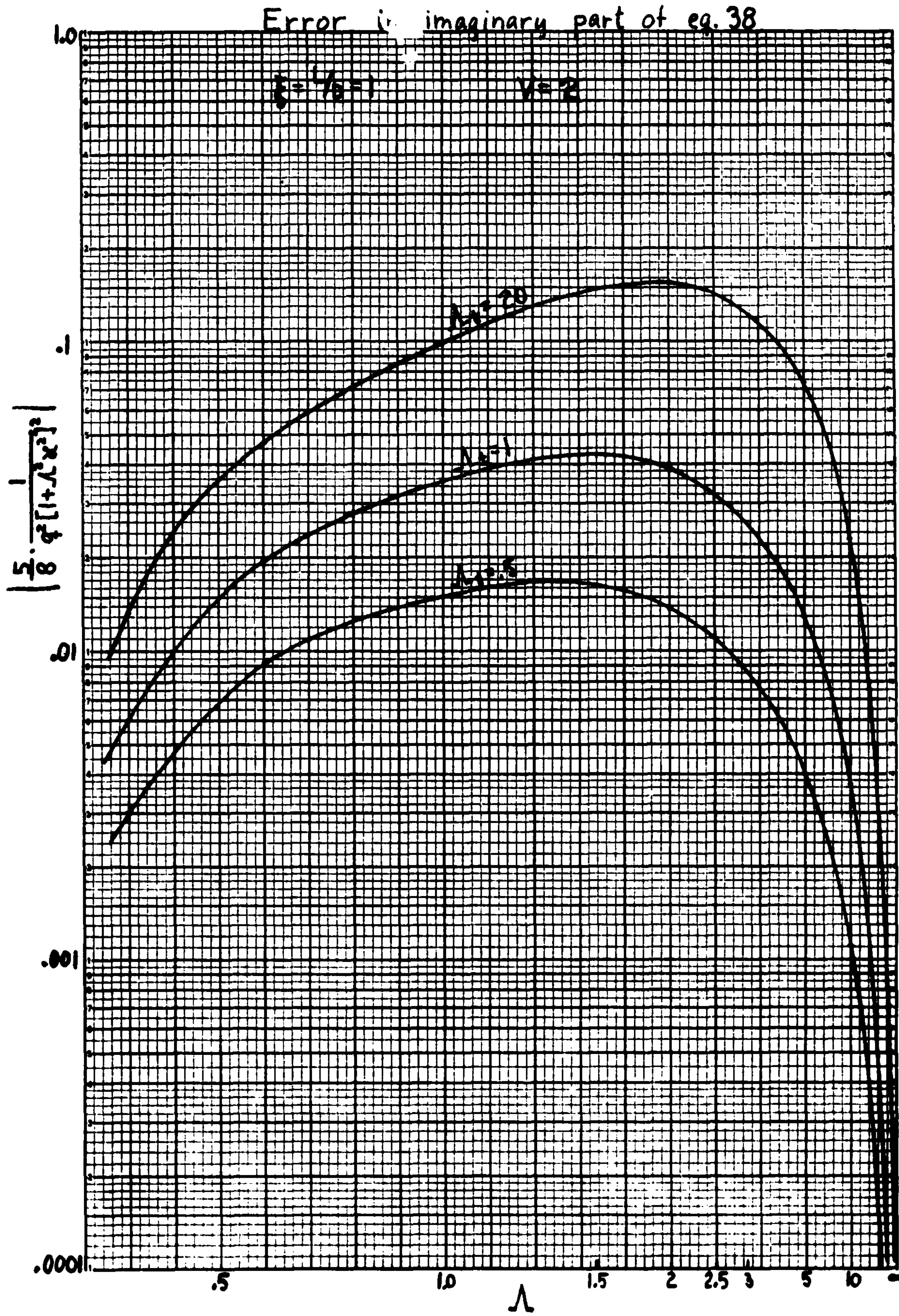
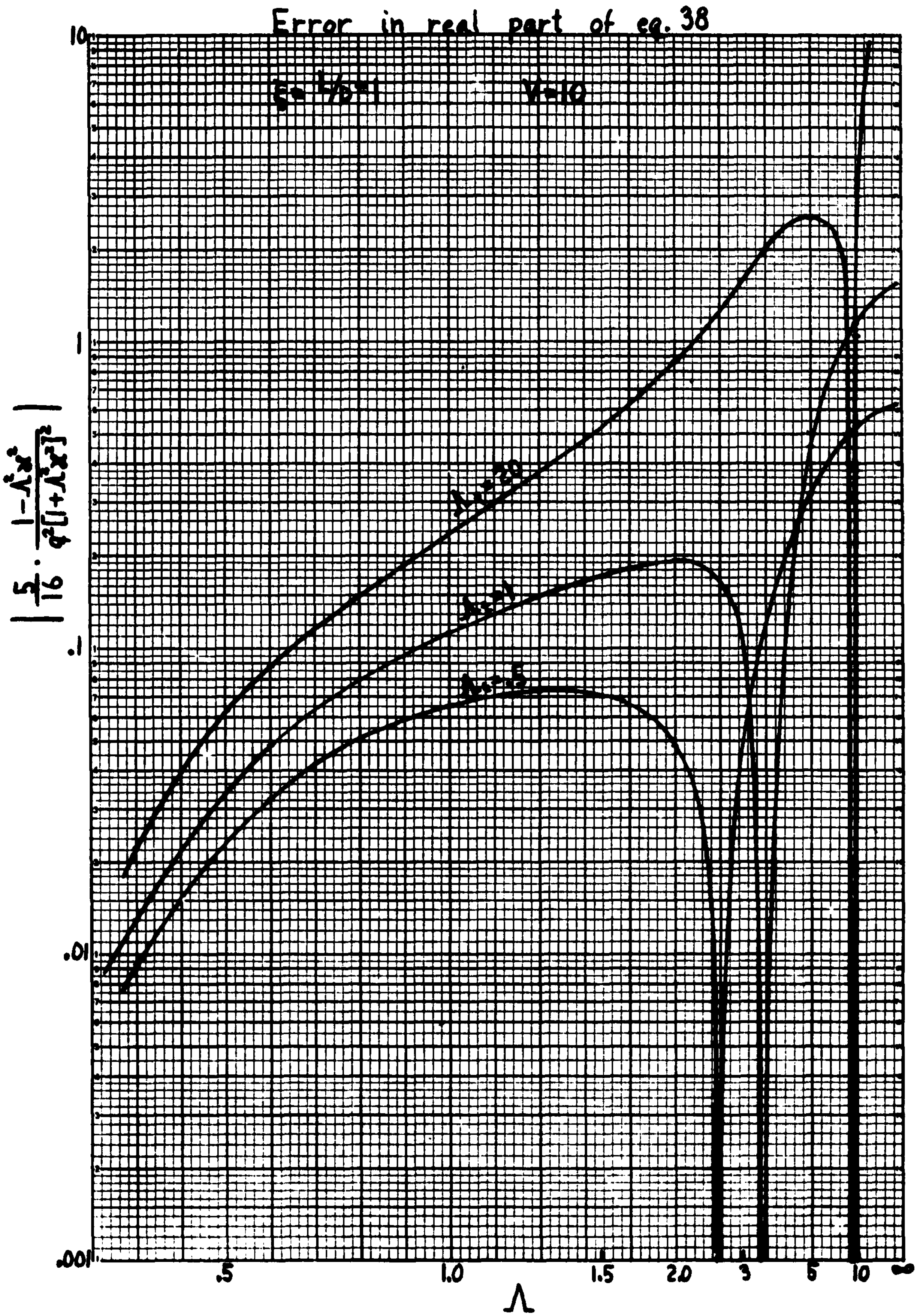


FIG. 14



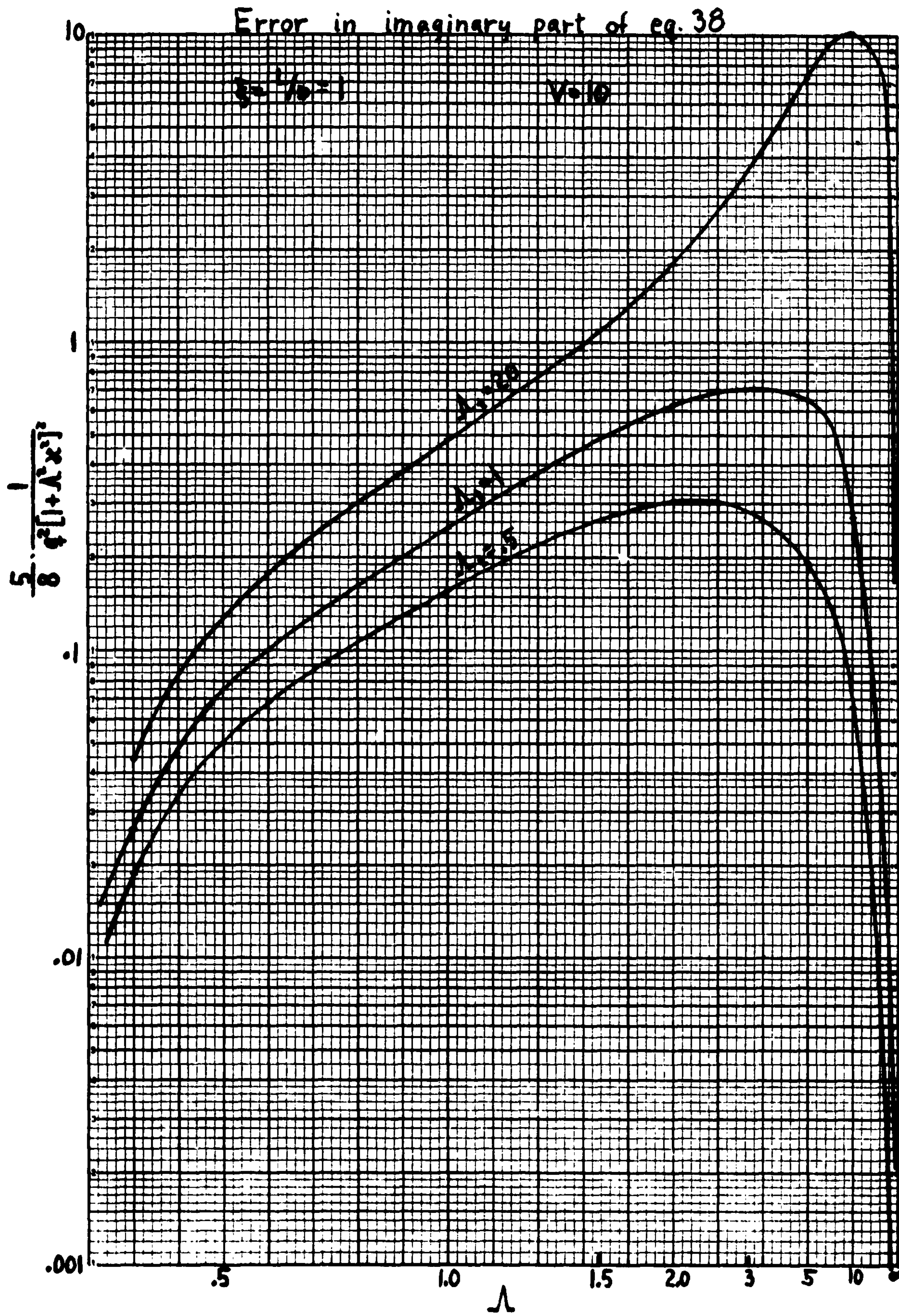
Error in imaginary part of eq. 38
$\frac{\xi}{\delta} = {}^{L}/_{D} = 1$
$V = 2$
$\Lambda_1 = 20$
$\Lambda_1 = 1$
$\Lambda_1 = .5$
$\left|\frac{5}{8} \cdot \frac{1}{q^2[1+\Lambda^2 x^2]^2}\right|$
1.0
.1
.01
.001
.0001
.5
1.0
1.5
2
2.5
3
5
10
∞
Λ

FIG. 15



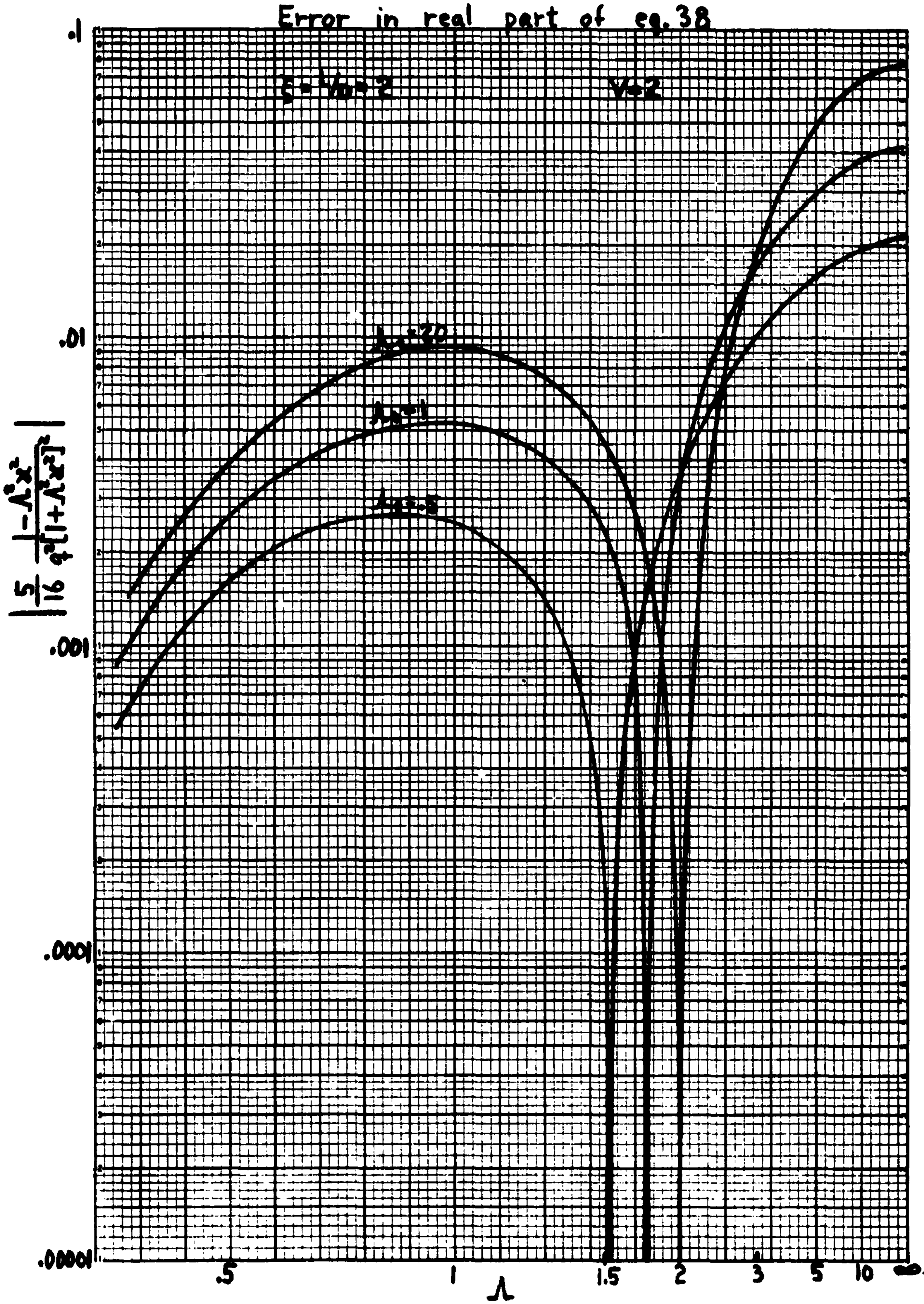
Error in real part of eq. 38
$\xi = L/D = 1$
$V = 10$
$\Lambda_c = 20$
$\Lambda_c = 1$
$\Lambda_c = .5$
10
1
.1
.01
.001
$\left| \frac{5}{16} \cdot \frac{1-\Lambda^2 x^2}{q^2[1+\Lambda^2 x^2]^2} \right|$
.5
1.0
1.5
2.0
3
5
10
∞
Λ

FIG. 16



Error in imaginary part of eq. 38
V=10
10
1
.1
.01
.001
.5
1.0
1.5
2.0
3
5
10
∞
Λ

FIG. 17



Error in real part of eq. 38
ξ = ¹/ω = 2
V=2
Λ₀=20
Λ₀=1
Λ₀=.5
$\left|\frac{5}{16}\,\frac{1-\Lambda^2 x^2}{q^2[1+\Lambda^2 x^2]^2}\right|$
.1
.01
.001
.0001
.00001
.5
1
1.5
2
3
5
10
∞
Λ

FIG. 18

Error in imaginary part of eq. 38

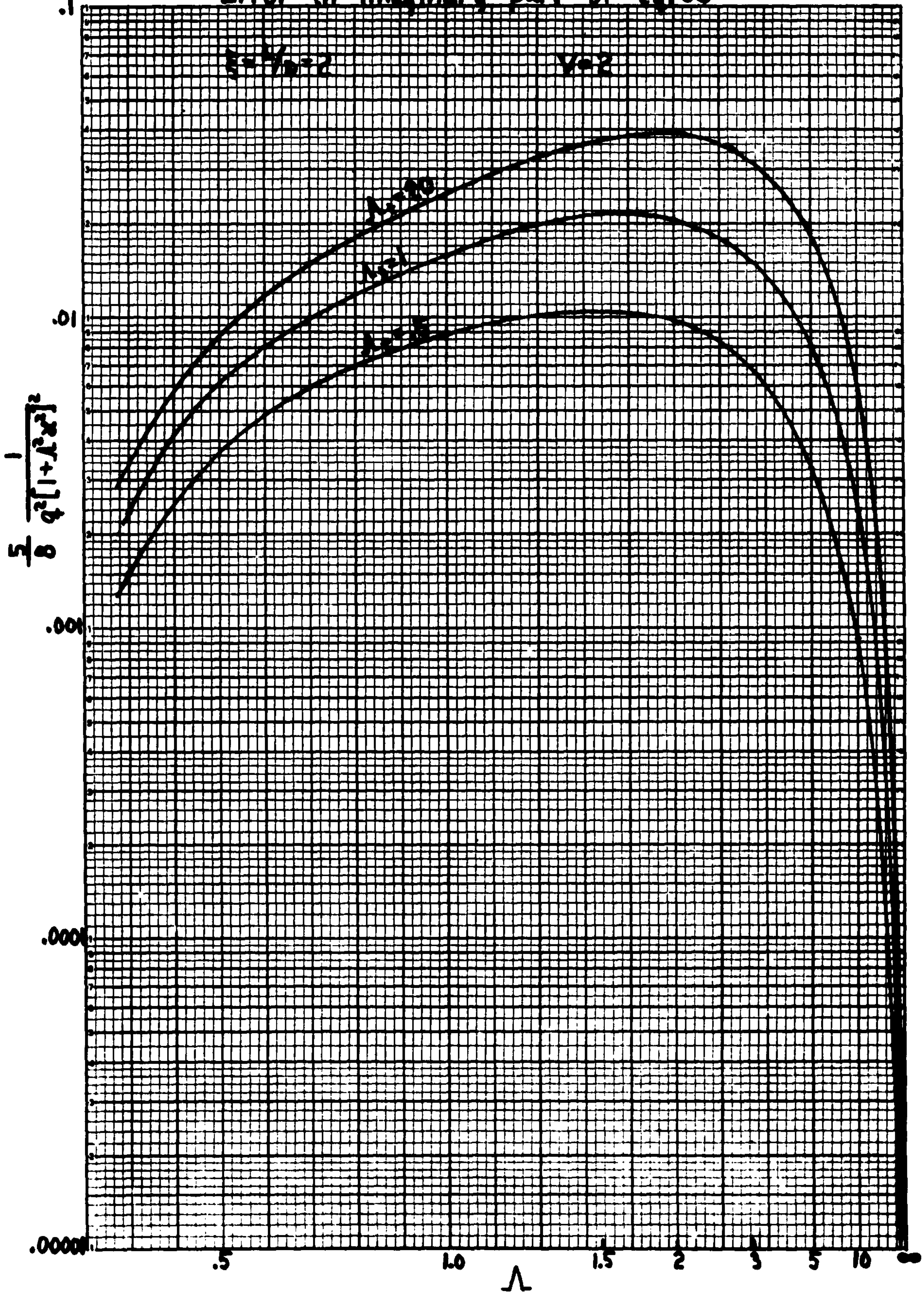
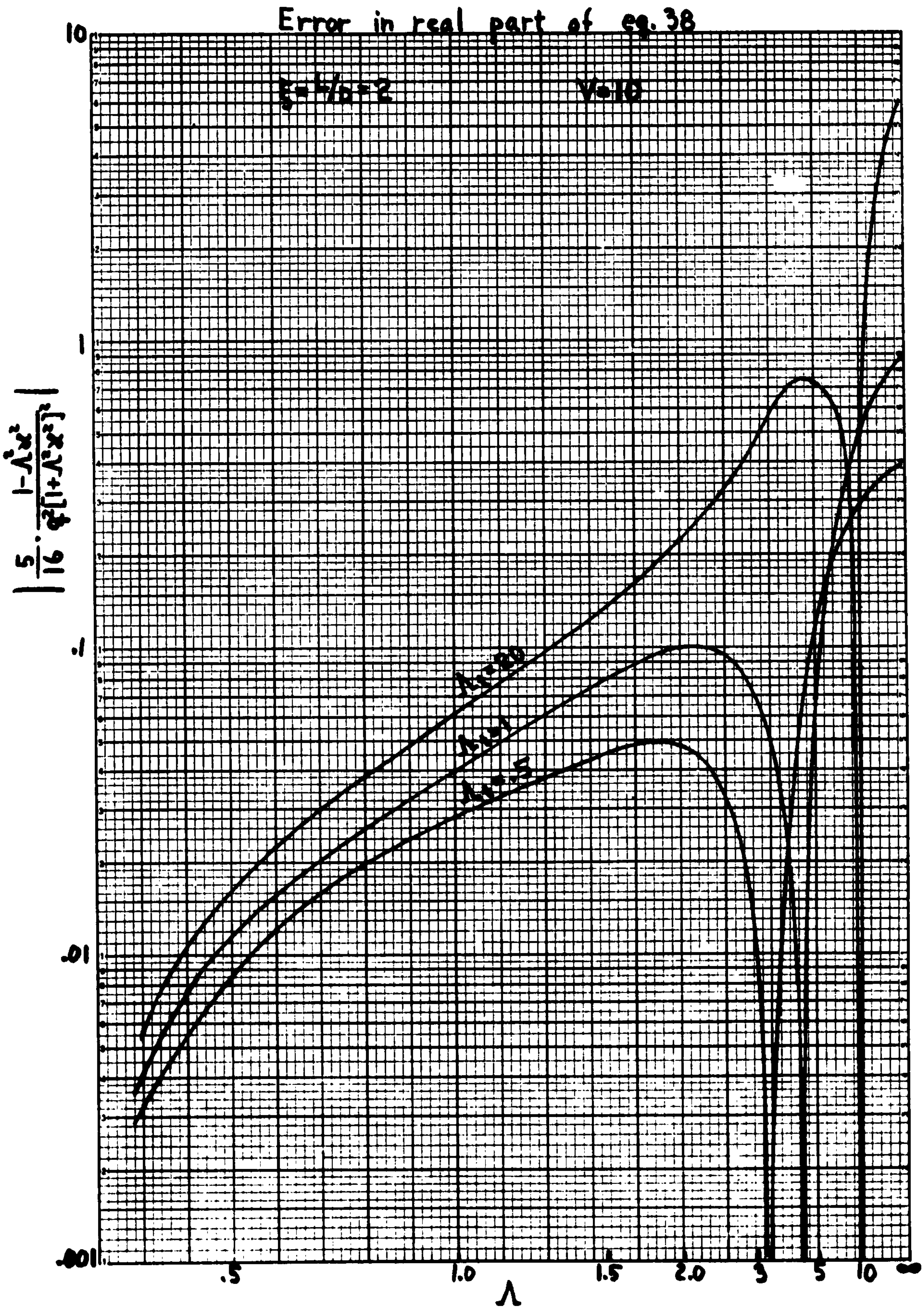
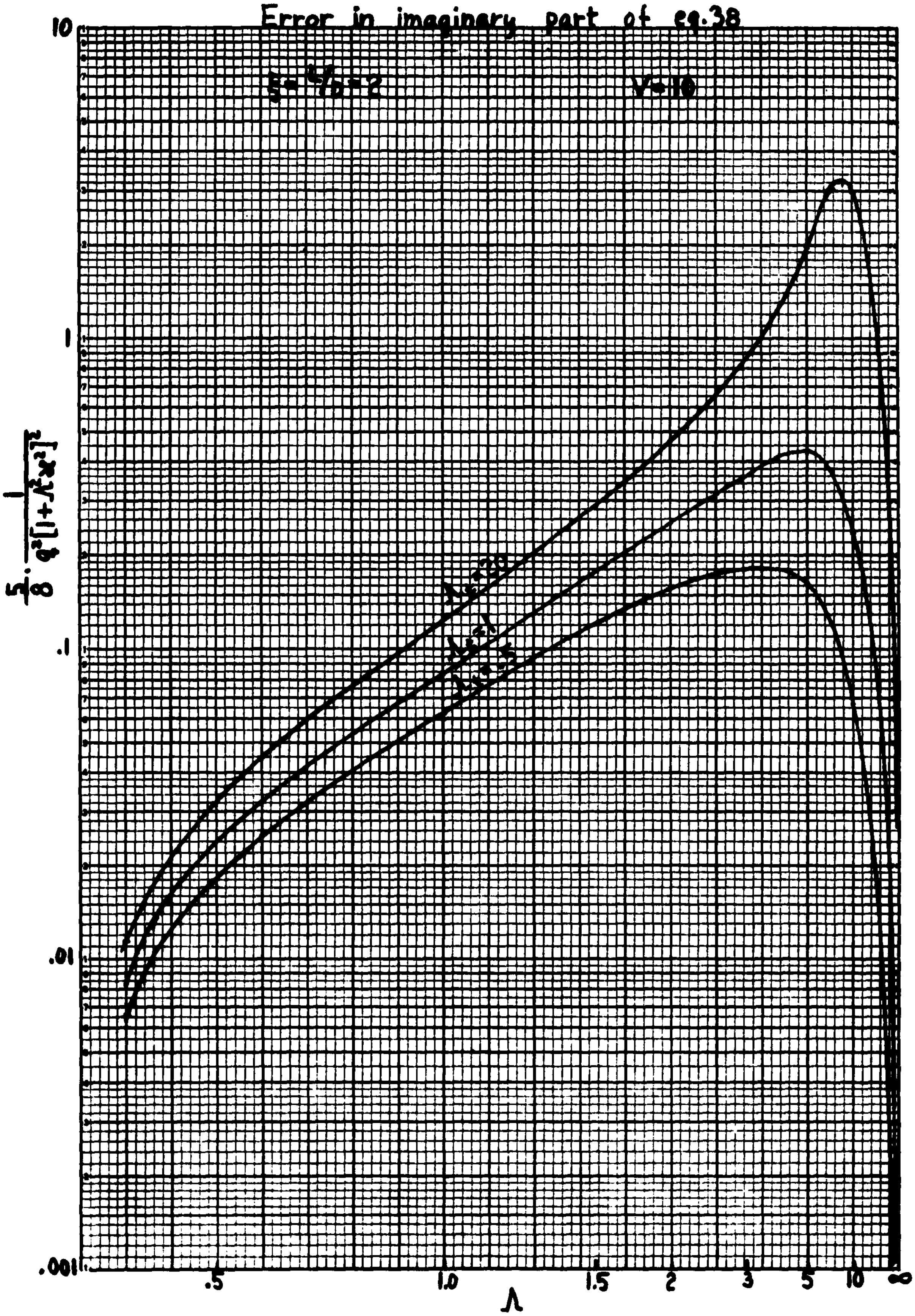


FIG. 19



Error in real part of eq. 38
$\xi = h/D = 2$
V=10
$\Lambda_1=20$
$\Lambda_1=1$
$\Lambda_1=.5$
10
1
.1
.01
.001
.5
1.0
1.5
2.0
3
5
10
∞
Λ
$\left|\frac{5}{16}\cdot\frac{1-\Lambda^2 x^2}{\xi^2[1+\Lambda^2 x^2]^2}\right|$

FIG. 20



Error in imaginary part of eq.38
$\xi = {}^{a}/_{b} = 2$
$V = 10$
$\frac{5}{8} \cdot \frac{1}{q^2[1+\Lambda^2 x^2]^2}$
10
1
.1
.01
.001
.5
1.0
1.5
2
3
5
10
∞
Λ
$\Lambda_c = 20$
$\Lambda_c = 1$
$\Lambda_c = .5$

FIG. 21

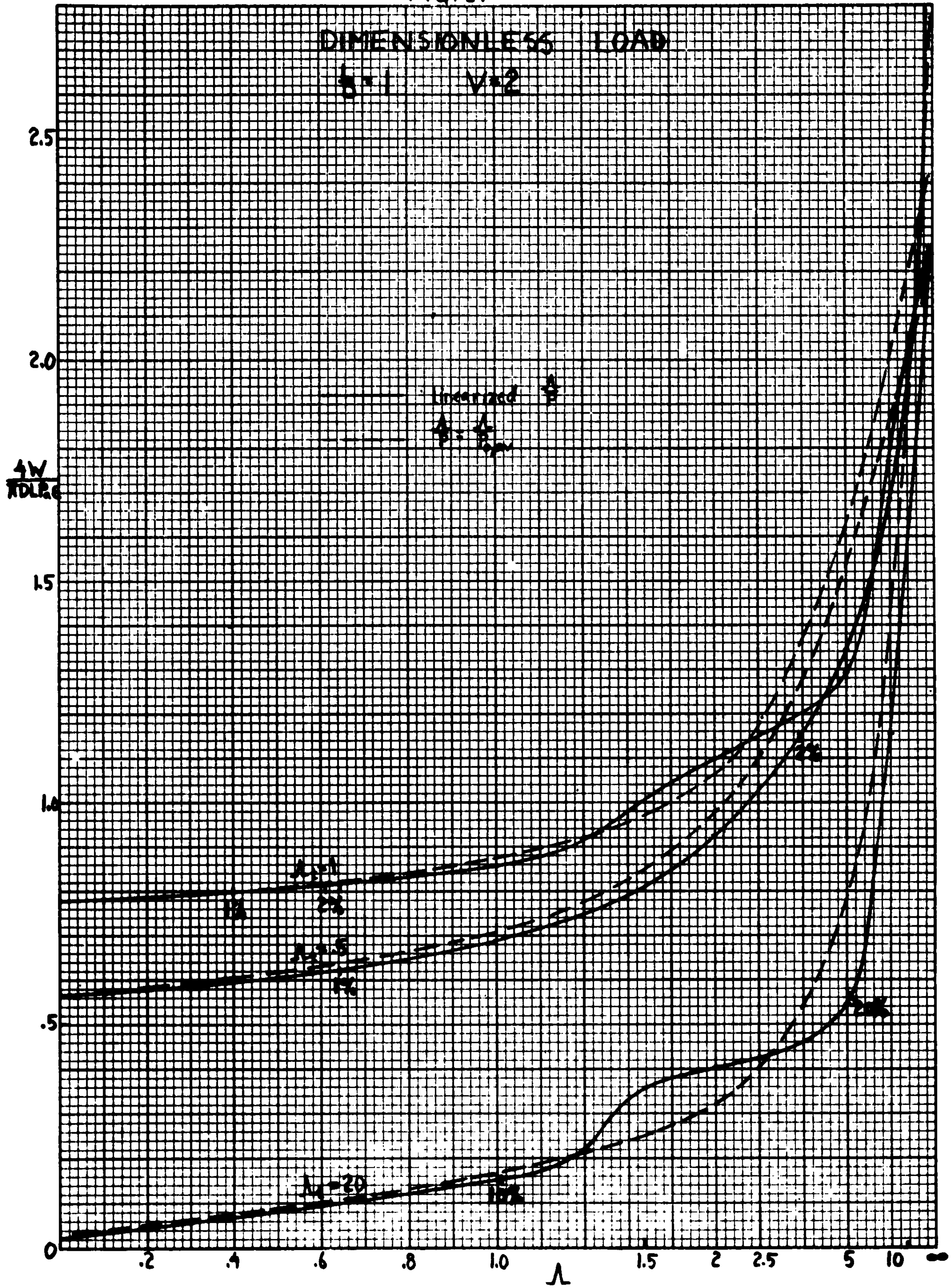
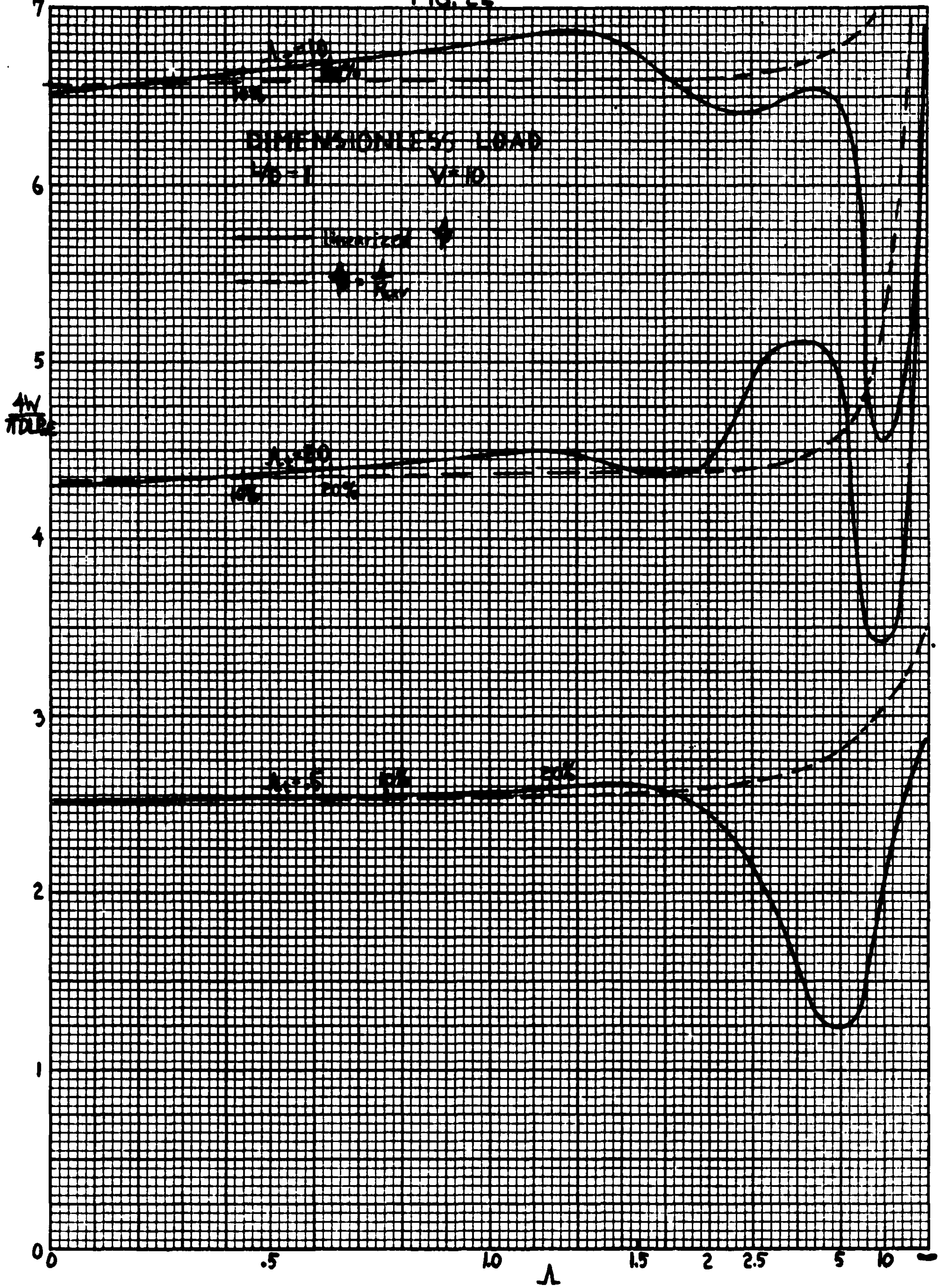

DIMENSIONLESS LOAD
V=2
Linearized
2.5
2.0
1.5
1.0
.5
0
.2
.4
.6
.8
1.0
1.5
2
2.5
5
10
∞
Λ

FIG. 22



DIMENSIONLESS LOAD
L/D = 1
V = 10
Linearized
$\Lambda_c = 10$
$\Lambda_c = 20$
$\Lambda_c = 5$
10%
20%
$\frac{4W}{\pi D L p_a}$
Λ
0
.5
1.0
1.5
2
2.5
5
10
∞
1
2
3
4
5
6
7

FIG. 23

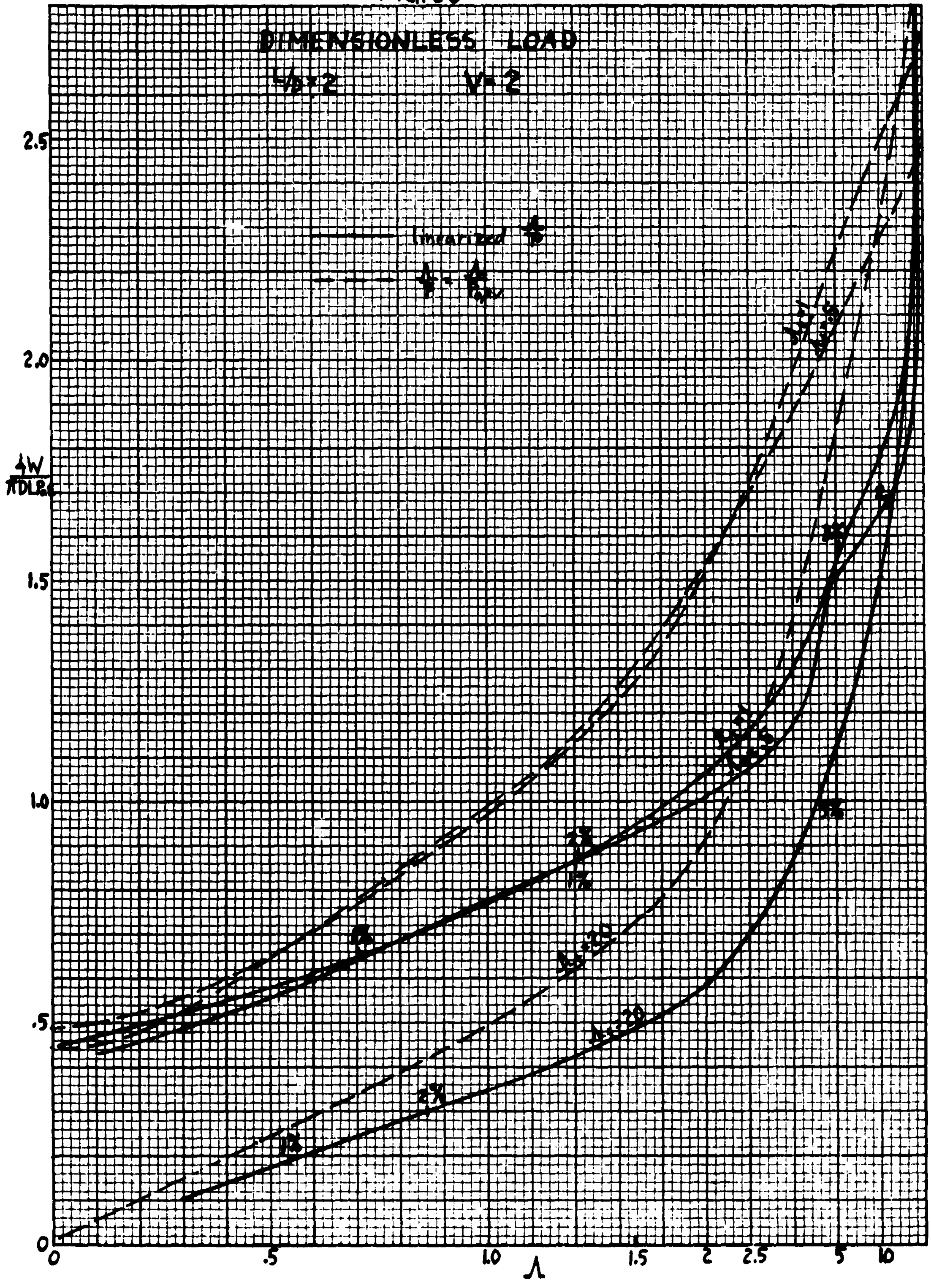
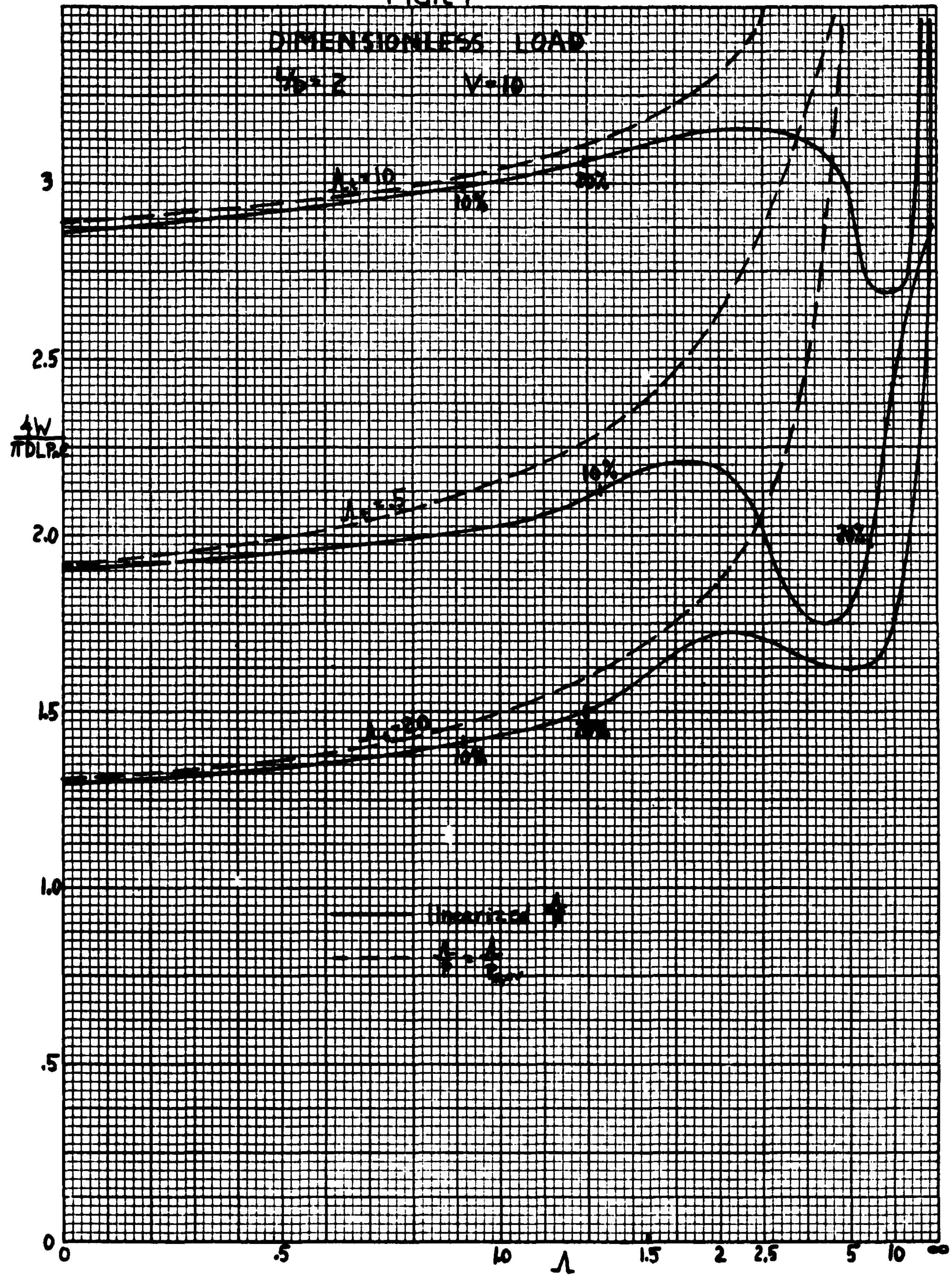

DIMENSIONLESS LOAD
L/D=2
V=2
linearized
2.5
2.0
1.5
1.0
.5
0
.5
1.0
1.5
2
2.5
5
10
Λ

FIG. 24



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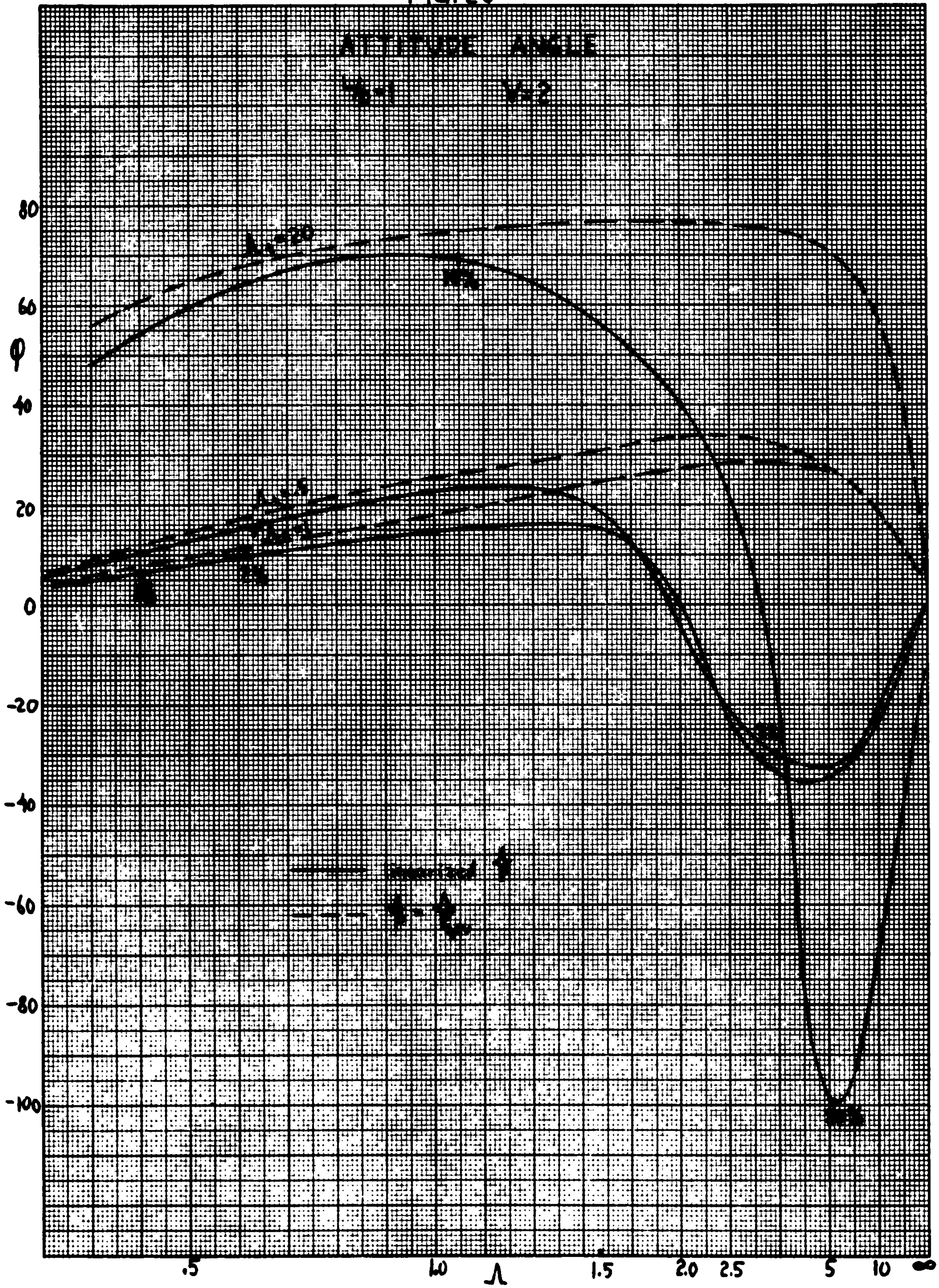


FIG. 26

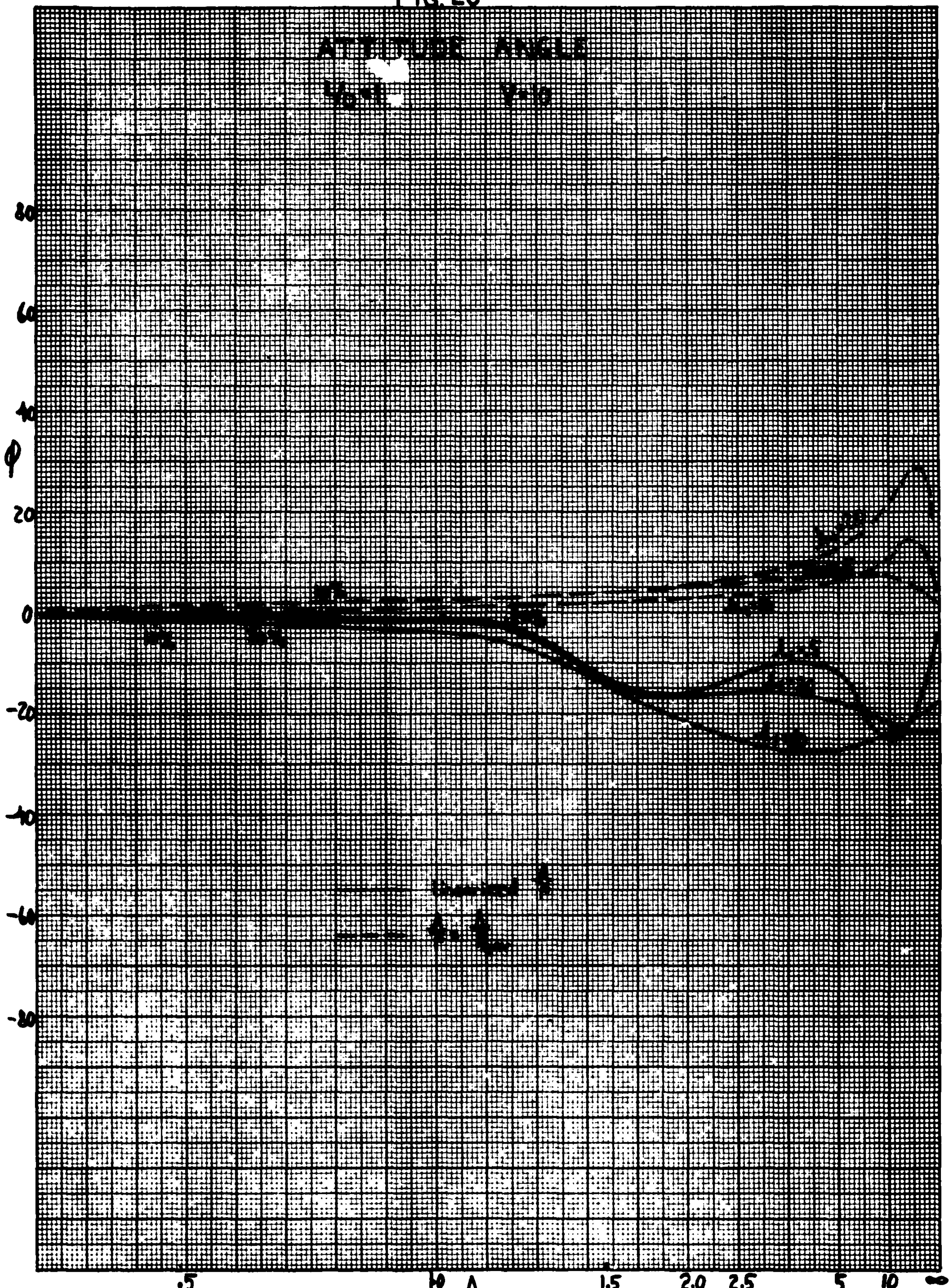


FIG. 27

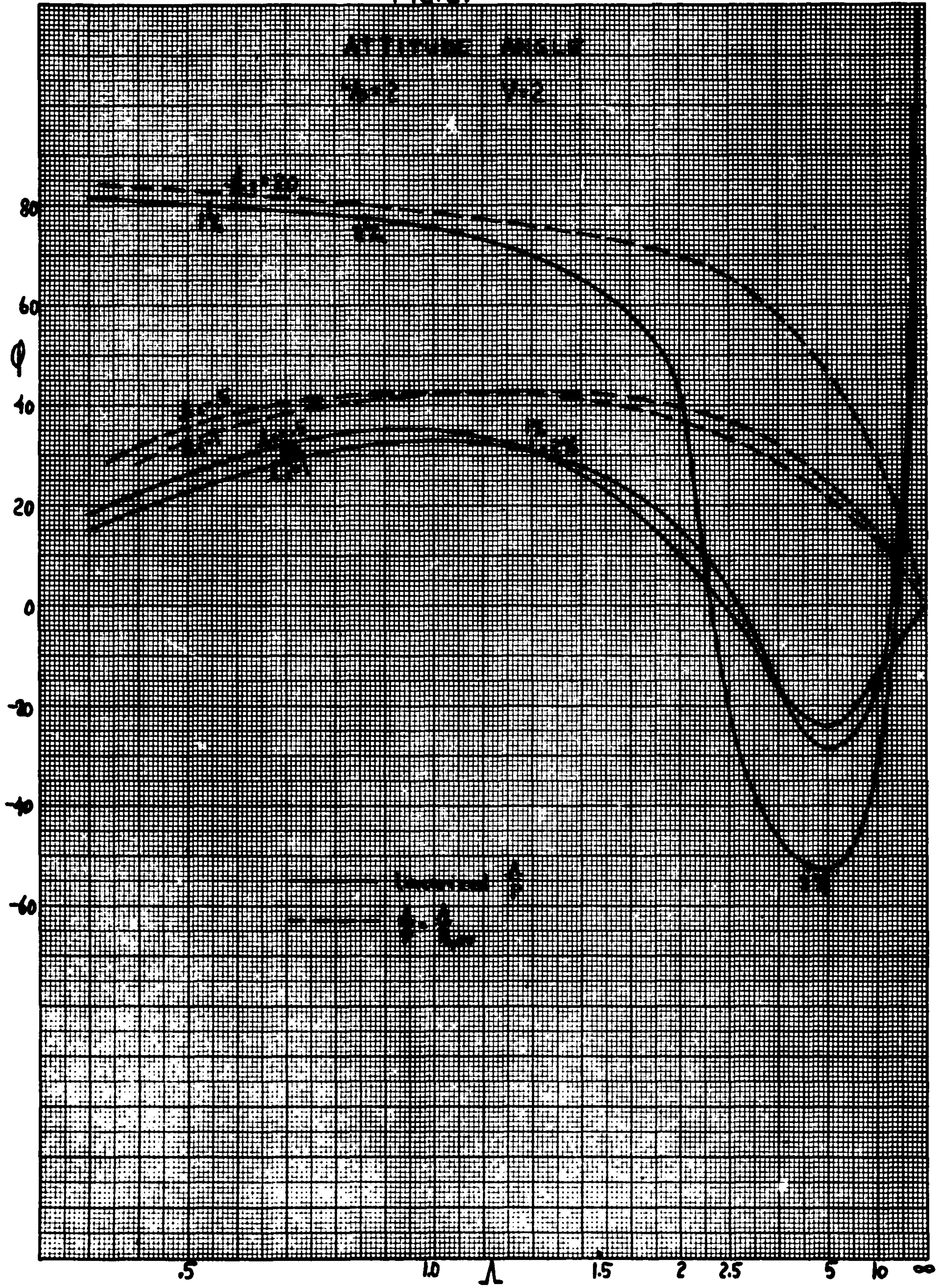


FIG. 28

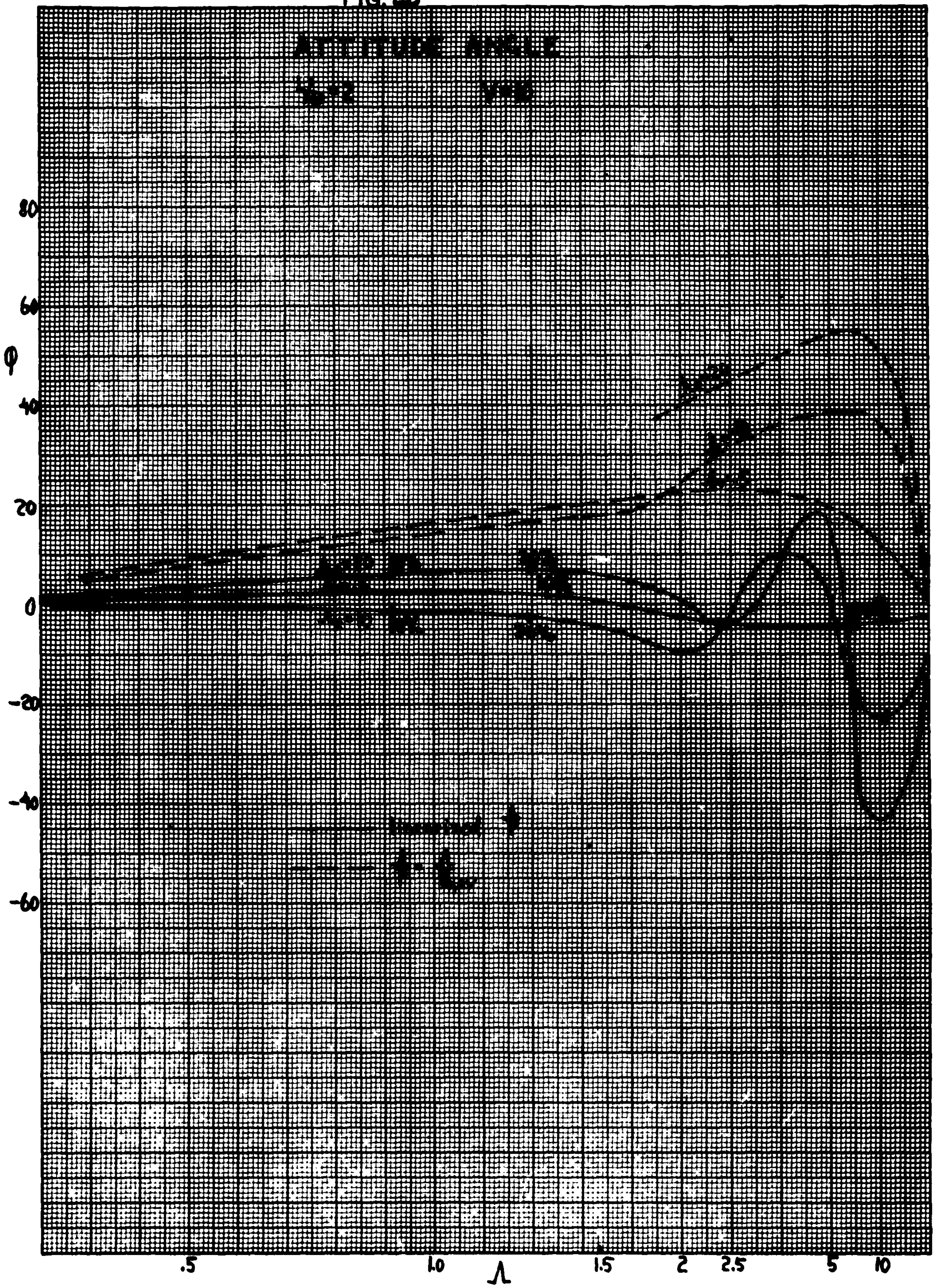


FIG. 29

DIMENSIONLESS FORCE

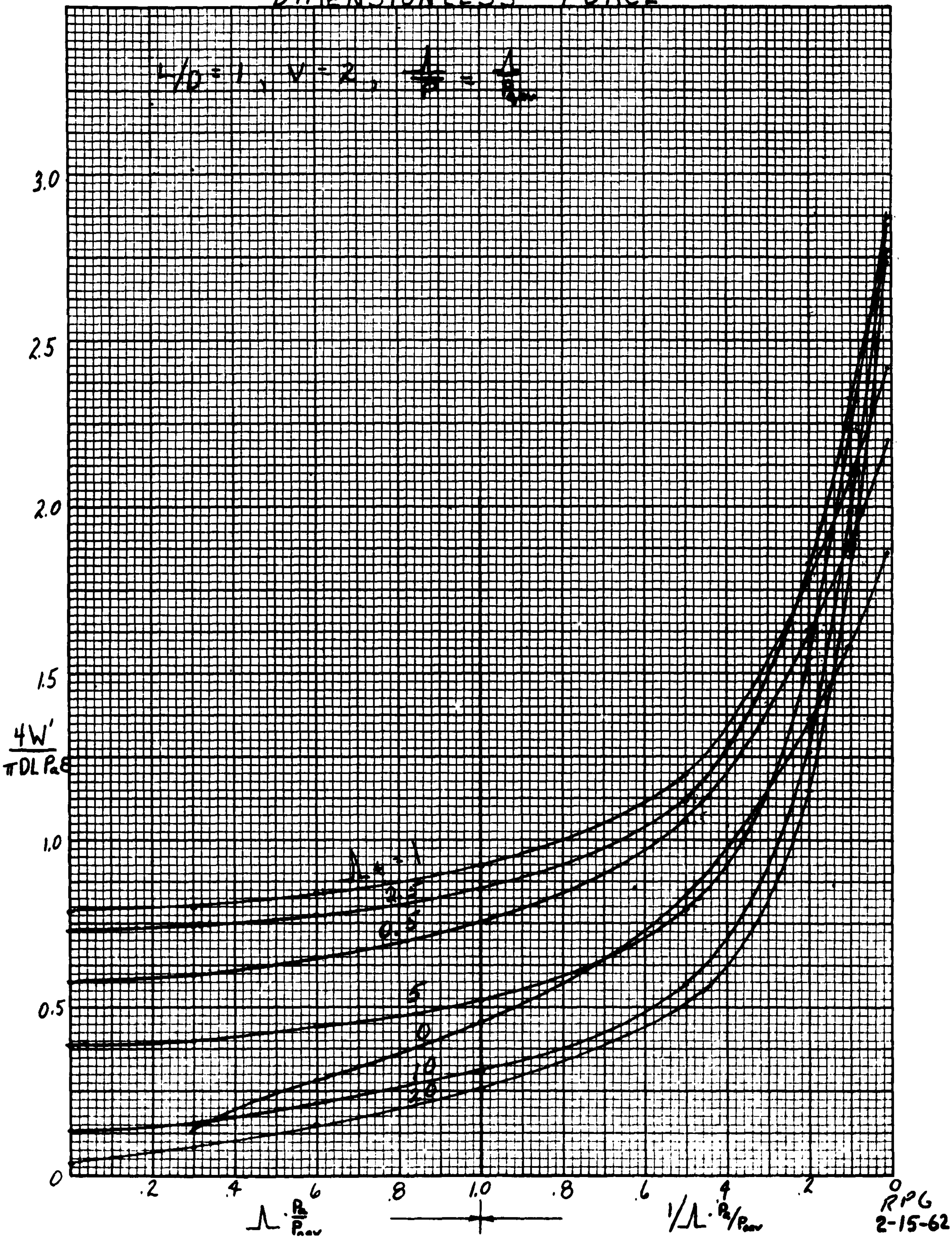


FIG. 30

DIMENSIONLESS FORCE

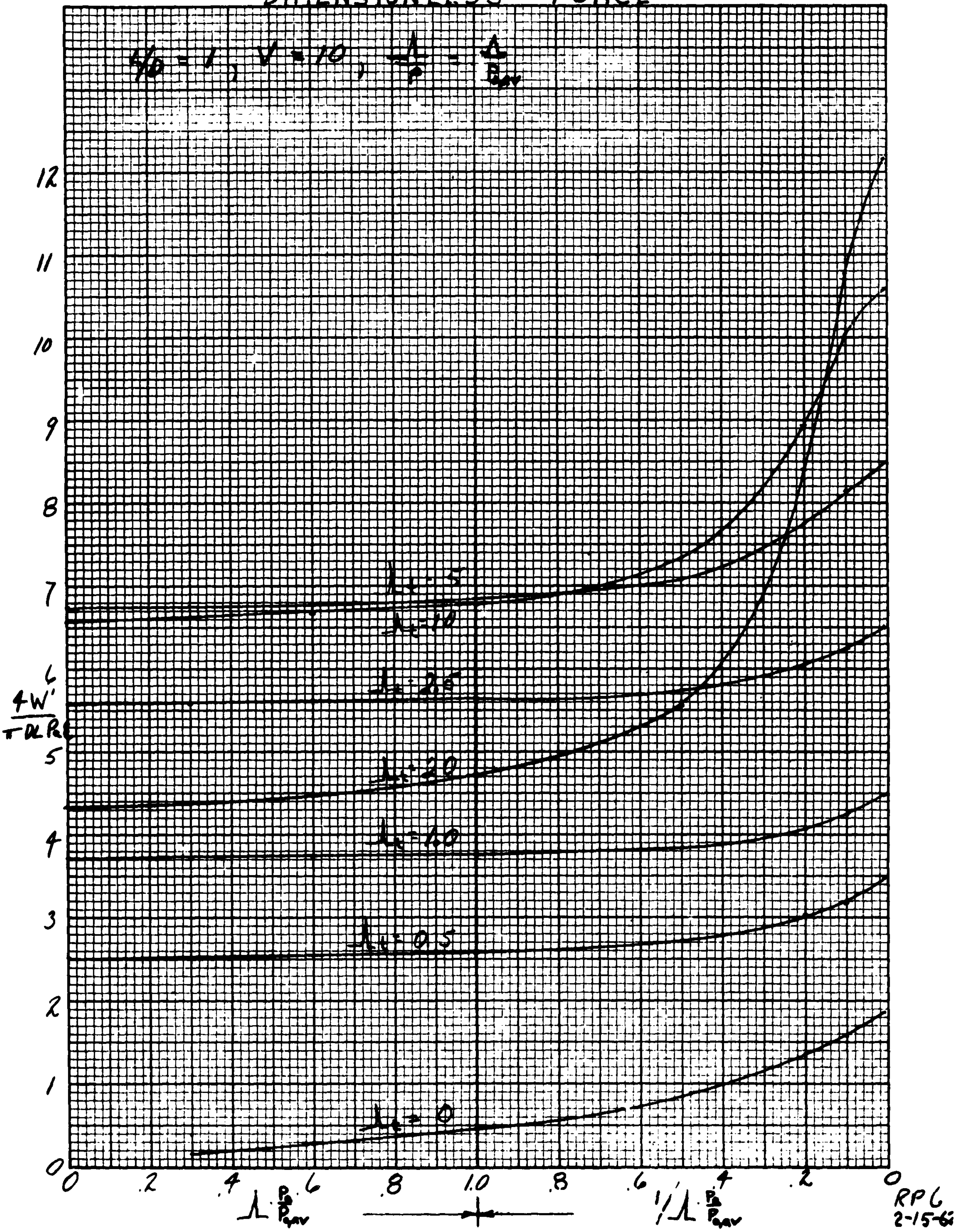


FIG. 31

DIMENSIONLESS FORCE

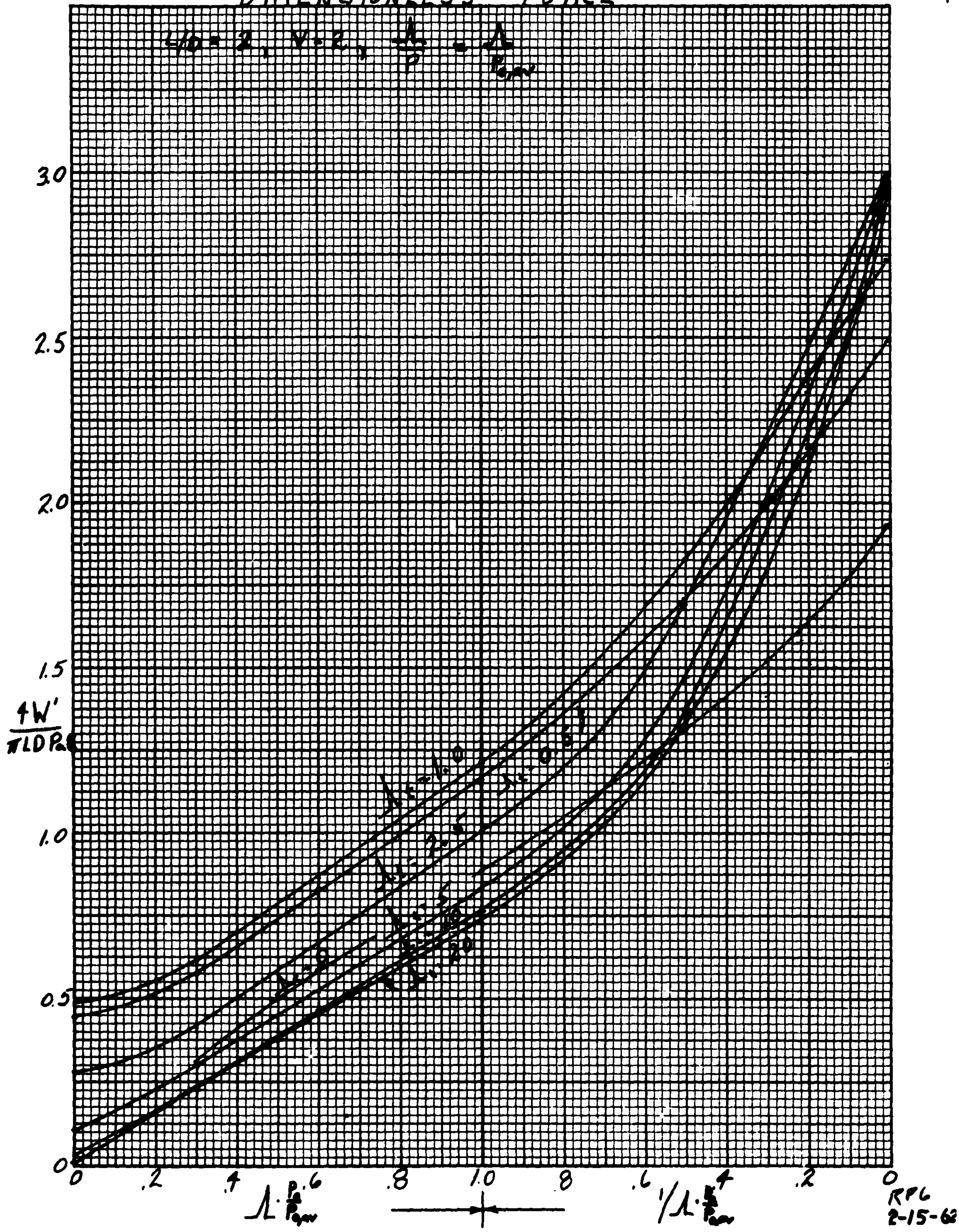
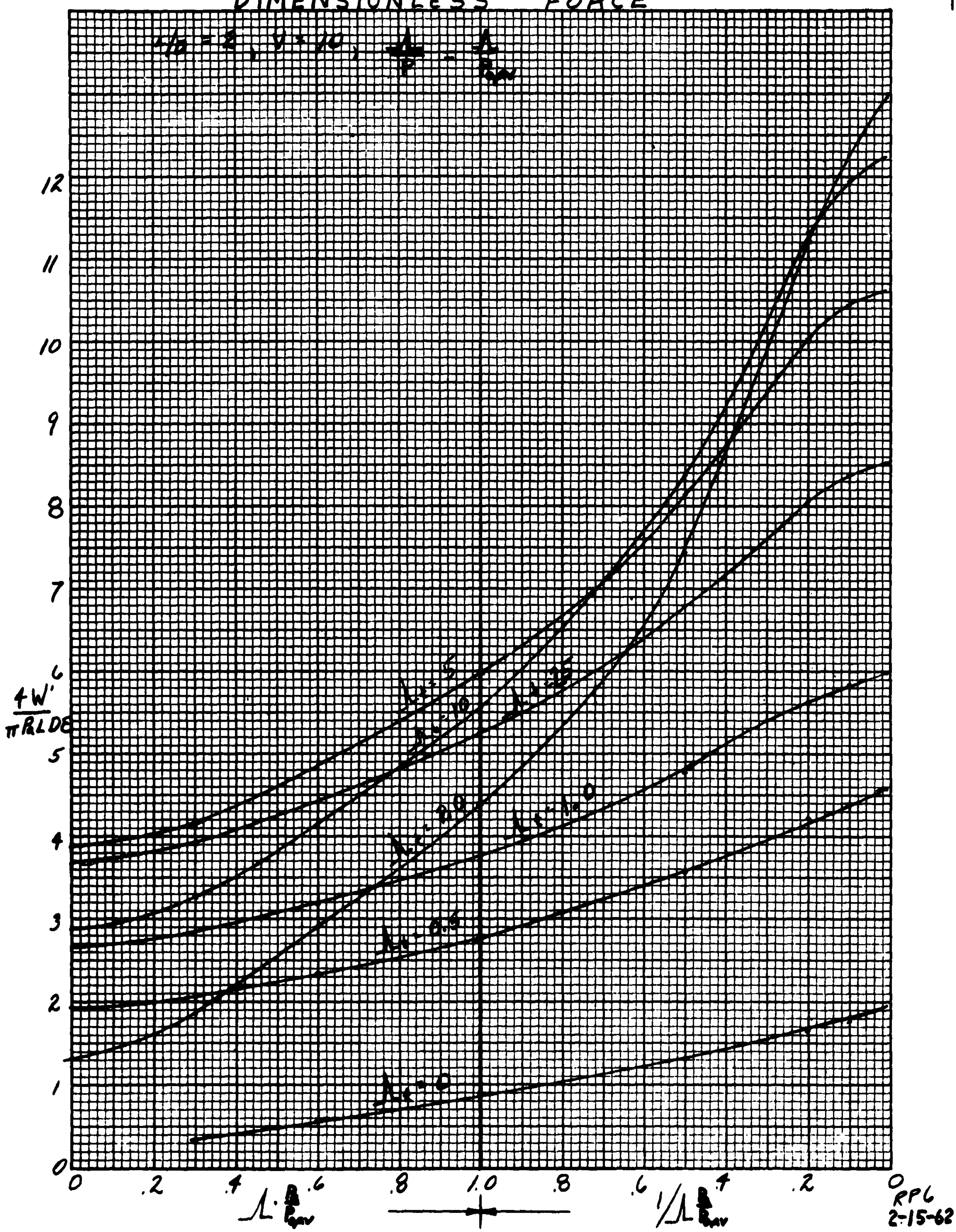


FIG. 32

DIMENSIONLESS FORCE



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